1. **Dieterici’s equation:** A gas obeys Dieterici’s equation of state:

\[ P(v - b) = k_B T \exp \left( -\frac{a}{k_B T v} \right), \]

where \( v = V/N \).

(a) Find the ratio \( P v / k_B T \) at the critical point.

(b) Calculate the isothermal compressibility \( \kappa_T \) for \( v = v_c \) as a function of \( T - T_c \).

(c) On the critical isotherm expand the pressure to the lowest non-zero order in \( (v - v_c) \).

2. **2d Coulomb Gas:** Consider a classical mixture of \( N \) positive, and \( N \) negative charged particles in a *two dimensional* box of area \( A = L \times L \). The Hamiltonian is

\[ \mathcal{H} = \sum_{i=1}^{2N} \frac{\vec{p}_i^2}{2m} - \sum_{i<j}^{2N} c_i c_j \ln |\vec{q}_i - \vec{q}_j|, \]

where \( c_i = +c_0 \) for \( i = 1, \ldots, N \), and \( c_i = -c_0 \) for \( i = N + 1, \ldots, 2N \), denote the charges of the particles; \( \{\vec{q}_i\} \) and \( \{\vec{p}_i\} \) their coordinates and momenta respectively.

(a) Note that in the interaction term each pair appears only once, and that there is no self interaction \( i = j \). How many pairs have repulsive interactions, and how many have attractive interactions?

(b) Write down the expression for the partition function \( Z(N, T, A) \) in terms of integrals over \( \{\vec{q}_i\} \) and \( \{\vec{p}_i\} \). Perform the integrals over the momenta, and rewrite the contribution of the coordinates as a product involving powers of \( \{\vec{q}_i\} \), using the identity \( e^{\ln x} = x \).

(c) Although it is not possible to perform the integrals over \( \{\vec{q}_i\} \) exactly, the dependence of \( Z \) on \( A \) can be obtained by the simple rescaling of coordinates, \( \vec{q}_i' = \vec{q}_i/L \). Use the results in parts (a) and (b) to show that \( Z \propto A^{2N - \beta c_0^2 N/2} \).

(d) Calculate the two dimensional pressure of this gas, and comment on its behavior at high and low temperatures.

(e) The unphysical behavior at low temperatures is avoided by adding a hard-core which prevents the coordinates of any two particles from coming closer than a distance \( a \). The appearance of two length scales \( a \) and \( L \), makes the scaling analysis of part (c) questionable.
By examining the partition function for \( N = 1 \), obtain an estimate for the temperature \( T_c \) at which the short distance scale \( a \) becomes important in calculating the partition function, invalidating the result of part (c). What are the phases of this system at low and high temperatures?

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3. **One dimensional gas:** In statistical mechanics, there are very few systems of interacting particles that can be solved exactly. Such exact solutions are very important as they provide a check for the reliability of various approximations. A one dimensional gas with short-range interactions is one such solvable case.

(a) Show that for a potential with a hard core that screens the interactions from further neighbors, the Hamiltonian for \( N \) particles can be written as

\[
\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i=2}^{N} V(x_i - x_{i-1}).
\]

The (indistinguishable) particles are labelled with coordinates \( \{x_i\} \) such that

\[
0 \leq x_1 \leq x_2 \leq \cdots \leq x_N \leq L,
\]

where \( L \) is the length of the box confining the particles.

(b) Write the expression for the partition function \( Z(T, N, L) \). Change variables to \( \delta_1 = x_1, \delta_2 = x_2 - x_1, \cdots, \delta_N = x_N - x_{N-1} \), and carefully indicate the allowed ranges of integration and the constraints.

(c) Consider the Gibbs partition function obtained from the Laplace transformation

\[
Z(T, N, P) = \int_{0}^{\infty} dL \exp(-\beta PL) Z(T, N, L),
\]

and by extremizing the integrand find the standard formula for \( P \) in the canonical ensemble.

(d) Change variables from \( L \) to \( \delta_{N+1} = L - \sum_{i=1}^{N} \delta_i \), and find the expression for \( Z(T, N, P) \) as a product over one-dimensional integrals over each \( \delta_i \).

(e) At a fixed pressure \( P \), find expressions for the mean length \( L(T, N, P) \), and the density \( n = N/L(T, N, P) \) (involving ratios of integrals which should be easy to interpret).

Since the expression for \( n(T, P) \) in part (e) is continuous and non-singular for any choice of potential, there is in fact no condensation transition for the one-dimensional gas. By contrast, the approximate van der Waals equation (or the mean-field treatment) incorrectly predicts such a transition.

(f) **(Optional)** For a hard sphere gas (as in PS#6, problem 3, or PS# 8, problem 2, part (d)) calculate the equation of state \( P(T, n) \). Compare the excluded volume factor with the approximate result obtained in PS#3, and also obtain the general virial coefficient \( B_{\ell}(T) \).

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**Suggested reading:** Feynman, Chapter 4; Ma, Chapter 8.