1. **Characteristic Functions:** Calculate the characteristic function, the mean, and the variance of the following probability density functions:

(a) Uniform \( p(x) = \frac{1}{2a} \) for \(-a < x < a\), and \( p(x) = 0 \) otherwise;

(b) Laplace \( p(x) = \frac{1}{2a} \exp\left(-\frac{|x|}{a}\right) \);

(c) Cauchy \( p(x) = \frac{a}{\pi(x^2 + a^2)} \).

The following two probability density functions are defined for \( x \geq 0 \). Compute only the mean and variance for each.

(d) Rayleigh \( p(x) = \frac{x}{a^2} \exp\left(-\frac{x^2}{2a^2}\right) \),

(e) Maxwell \( p(x) = \sqrt{\frac{2}{\pi a^3}} \exp\left(-\frac{x^2}{2a^2}\right) \).

2. **Directed Random Walk:** The motion of a particle in three dimensions is a series of independent steps of length \( \ell \). Each step makes an angle \( \theta \) with the \( z \) axis, with a probability density \( p(\theta) = 2\cos^2(\theta/2)/\pi \); while the polar angle \( \phi \) is uniformly distributed between 0 and 2\( \pi \). (Note that the solid angle factor of \( \sin \theta \) is already included in the definition of \( p(\theta) \), which is correctly normalized to unity.) The particle (walker) starts at the origin and makes a large number of steps \( N \).

(a) Calculate the expectation values \( \langle z \rangle \), \( \langle x \rangle \), \( \langle y \rangle \), \( \langle z^2 \rangle \), \( \langle x^2 \rangle \), and \( \langle y^2 \rangle \), and the covariances \( \langle xy \rangle \), \( \langle xz \rangle \), and \( \langle yz \rangle \).

(b) Use the central limit theorem to estimate the probability density \( p(x, y, z) \) for the particle to end up at the point \( (x, y, z) \).

3. **Tchebycheff’s Inequality:** Consider any probability density \( p(x) \) for \((-\infty < x < \infty\)\), with mean \( \lambda \), and variance \( \sigma^2 \). Show that the total probability of outcomes that are more than \( n\sigma \) away from \( \lambda \) is less than \( 1/n^2 \), i.e.

\[
\int_{|x-\lambda|\geq n\sigma} dxp(x) \leq \frac{1}{n^2}.
\]

**Hint:** Start with the integral defining \( \sigma^2 \), and break it up into parts corresponding to \( |x-\lambda| > n\sigma \), and \( |x-\lambda| < n\sigma \).
4. **Optimal Selections:** In many specialized populations, there is little variability among the members (e.g., in the GRE scores of the 8.333 students compared to GRE scores of a random group.) Is this a natural consequence of optimal selection?

(a) Let \( \{r_\alpha\} \) be \( n \) random numbers, each independently chosen from a probability density \( p(r) \), with \( r \in [0, 1] \). Calculate the probability density \( p_n(x) \) for the largest value of this set, i.e. for \( x = \max\{r_1, \cdots, r_n\} \).

(b) If each \( r_\alpha \) is uniformly distributed between 0 and 1, calculate the mean and variance of \( x \) as a function of \( n \), and comment on their behavior at large \( n \).

5. **Information:** Consider the velocity of a gas particle in one dimension \((-\infty < v < \infty)\).

(a) Find the unbiased probability density \( p_1(v) \), subject only to the constraint that the average speed is \( c \), i.e. \( \langle |v| \rangle = c \).

(b) Now find the probability density \( p_2(v) \), given only the constraint of average kinetic energy, \( \langle mv^2/2 \rangle = mc^2/2 \).

(c) Which of the above statements provides more information on the velocity? Quantify the difference in information in terms of \( I_2 - I_1 \equiv (\langle \ln p_2 \rangle - \langle \ln p_1 \rangle) / \ln 2 \).

(Optional) 6. **Benford’s Law** describes the observed probabilities of the first digit in a great variety of data sets, such as stock prices. Rather counter-intuitively, the digits 1 through 9 occur with probabilities 0.301, .176, .125, .097, .079, .067, .058, .051, .046 respectively. The key observation is that this distribution is invariant under a change of scale, e.g. if the stock prices were converted from dollars to persian rials! Find a formula that fits the above probabilities on the basis of this observation. (*Hint:* Think about random multiplicative processes.)

*Suggested Reading:* S.-K. Ma, Statistical Mechanics, Part III.