1. **Continuity equation for probability density and probability current** (20 points)

A particle is in a state described by a wavefunction $\Psi(x,t)$.

a) (10 points) Using the Schrödinger equation, show that the probability density $P(x,t) = |\Psi(x,t)|^2$ obeys the continuity equation

$$\frac{\partial}{\partial t} P(x,t) + \frac{\partial}{\partial x} J(x,t) = 0,$$

if we define the probability current $J(x,t)$ by

$$J(x,t) = \frac{\hbar}{2im} \left( \Psi^*(x,t) \frac{\partial \Psi(x,t)}{\partial x} - \frac{\partial \Psi^*(x,t)}{\partial x} \Psi(x,t) \right).$$

b) (10 points) Write down an integral form of the continuity equation, i.e. a relation between the current $J$ as defined above and the probability $P_{ab}(t)$ of finding a particle inside a finite interval $a \leq x \leq b$. Express the physical meaning of this equation in one sentence.

2. **Fictitious Bohr atom.** (20 points)

What would the Balmer formula look like for a fictitious Bohr atom where the electron is bound to the nucleus by a potential $V(r) = -\frac{C_6}{r^6}$? Use the Bohr quantization condition $L = n\hbar$ for the angular momentum for circular orbits to calculate the energy levels corresponding to different quantum number $n$, and remember which transitions the Balmer formula corresponds to. Find the quantity that would correspond to the Rydberg constant, and express it in terms of $C_6$, the electron mass $m$, and $\hbar$.

3. **Sommerfeld-Wilson quantization for linear potential in one dimension.** (25 points)

Consider a particle of mass $m$ linear potential $V(x) = C|x|$, $C>0$. We want to determine the quantized energy levels in such a potential.

a) Assume $x(t=0) = A$, $A > 0$, and $p(t=0) = 0$. Calculate $x(t)$ and $p(t)$ for one period $T$.

   How large is $T$?

b) Calculate $\int p(x) dx$, i.e. the integral over one period, as a function of $C$, particle mass $m$, and amplitude of motion $A$. Note that $dx < 0$ ($dx > 0$) if the particle moves towards negative (positive) $x$ values.

c) Now use the Sommerfeld-Wilson quantization condition $\int p(x) dx = n\hbar$ to determine a quantum mechanical condition on the amplitude $A_n$. What is the value of the amplitude for the ground state $A_1$, i.e. the amplitude for $n=1$?

d) Calculate the quantized energy levels $E_n$. Sketch the potential $V(x)$ and the quantized energy levels $E_n$. Compare the dependence of the spacing between energy levels on quantum number $n$ to the Bohr atom.

Plot the motion in phase space, i.e. in a momentum-versus-position diagram. What is the geometrical meaning of the Sommerfeld-Wilson quantization condition? In one sentence, how would you describe the stationary states in phase space?
4. **Momentum expectation values in terms of spatial wavefunctions.** (20 points)

We have defined the expectation value of a function $g(p)$ of momentum in terms of the probability density in wavevector (or momentum) space as

$$
\langle g(p) \rangle = \int_{-\infty}^{\infty} dk \, g(hk) \tilde{\phi}(k)^2 = \int_{-\infty}^{\infty} dp \, g(p) \phi(p)^2
$$

For $g(p)=p$, this is simply

$$
\langle p \rangle = \int_{-\infty}^{\infty} dk \, h k \tilde{\phi}(k)^2 = \int_{-\infty}^{\infty} dp \, p \phi(p)^2 dp
$$

Show that we can instead calculate the expectation value of momentum directly from the spatial wavefunction as

$$
\langle p \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right) \psi(x)
$$

and in general

$$
\langle p^n \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right)^n \psi(x).
$$

This means that momentum $p$ is represented in the spatial domain as an operator $\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$.

Show that similarly the expectation value of the position operator can be written as

$$
\langle x \rangle = \int_{-\infty}^{\infty} dp \, \phi^*(p) \left( i \hbar \frac{\partial}{\partial p} \right) \phi(p)
$$

and

$$
\langle x^n \rangle = \int_{-\infty}^{\infty} dp \, \phi^*(p) \left( i \hbar \frac{\partial}{\partial p} \right)^n \phi(p).
$$

What is the representation of the particle's position $x$ in the momentum domain?

5. **Relations for probability current.** (15 points)

a) (5 points) Show that the probability current can be written as

$$
J(x,t) = \frac{1}{2m} \left[ \Psi^*(x,t) \hat{p} \Psi(x,t) + (\Psi^*(x,t) \hat{p} \Psi(x,t))^* \right].
$$

b) (10 points) Show that a complex potential $V(x)^* \neq V(x)$ contradicts the continuity equation.