1. **Operators and the HO** (30 points)

Let \( |0\rangle \) be the normalized ground state of the harmonic oscillator, defined by \( \hat{a}|0\rangle = 0 \) and \( \langle 0|0\rangle = 1 \), where \( \hat{a} \) is the lowering (or annihilation) operator and \( \hat{a}^+ \) is the raising (or creation) operator.

a) (5 points) Show that the length of the unnormalized \( n \)-th eigenstate \( |n\rangle \), defined by 
\[
|n\rangle = (\hat{a}^+)^n |0\rangle,
\]
is given by 
\[
\sqrt{n}!.
\]

b) (5 points) As a consequence of a), the normalized \( n \)-th eigenstate \( |n\rangle \) is given by 
\[
|n\rangle = \frac{1}{\sqrt{n}!}(\hat{a}^+)^n |0\rangle.
\]
Verify that the eigenstates are orthonormal, i.e. show that \( \langle n|m\rangle = \delta_{nm} \).

c) (5 points) Show that 
\[
\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \text{and} \quad \hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle.
\]

d) (5 points) Define an operator \( \hat{n} \) by 
\[
\hat{n} = \hat{a}^+ \hat{a}.
\]
Show that \( \hat{n} \) is a Hermitian operator. Consequently, \( \hat{n} \) represents a measurable quantity. Calculate \( \langle n|\hat{n}|n\rangle \). What is the physical quantity that \( \hat{n} \) represents? What are the eigenvalues and eigenstates of \( \hat{n} \)?

e) (5 points) Express \( \hat{x} \) and \( \hat{p} \) in terms of \( \hat{a} \) and \( \hat{a}^+ \). Calculate \( \langle m|\hat{x}|n\rangle \) and \( \langle m|\hat{p}|n\rangle \), and show that they vanish unless \( m = n \pm 1 \).

f) (5 points) Express \( \hat{x}^2 \) and \( \hat{p}^2 \) in terms of \( \hat{a} \) and \( \hat{a}^+ \). Show that in the \( n \)-th eigenstate of the harmonic oscillator the uncertainty product of \( x \) and \( p \) is given by 
\[
\Delta x \Delta p = \frac{\hbar}{2}(2n + 1).
\]

2. **Coherent states of the HO** (40 points)

A state \( |\alpha\rangle \) that obeys the eigenequation \( \hat{a}|\alpha\rangle = \alpha|\alpha\rangle \) with an arbitrary complex number \( \alpha \), is called a coherent state.

a) (5 points) Show that 
\[
\hat{a}(\hat{a}^+)^n |0\rangle = n(\hat{a}^+)^n |0\rangle.
\]

b) (10 points) Show that the coherent state may be written in the form 
\[
|\alpha\rangle = C \exp(\alpha \hat{a}^+) |0\rangle,
\]
where \( C \) is a normalization constant. The operator \( \exp(\alpha \hat{a}^+) \) is defined in terms of its Taylor expansion.

c) (5 points) Calculate the normalization constant \( C \).

d) (10 points) Expand the coherent state in terms of the normalized eigenstates \( |n\rangle \), and calculate the probability of finding the HO in the \( n \)-th eigenstate (or equivalently, of finding \( n \) quanta in the system). You should obtain the Poisson distribution.

e) (10 points) Calculate the average number of quanta \( \langle n \rangle = \langle \alpha | \hat{n} | \alpha \rangle \) in the coherent state.

The coherent states are the closest quantum mechanical analog to classical states with a well-defined amplitude and phase. They are used, e.g., in the quantum mechanical description of the light field emitted
by a laser or radiofrequency oscillator. The coherent states are also the appropriate states for describing a classical harmonic oscillator with a large average quantum number $\langle n \rangle$. 

3. **Projection operator** (10 points)

Let $|n\rangle$ denote the eigenstate of a Hermitian operator. The operator $\hat{P}_n = |n\rangle\langle n|$ is called the projection operator. Show that $\hat{P}_m \hat{P}_n = \delta_{mn} \hat{P}_n$. Using the expansion of an arbitrary state into eigenstates $|n\rangle$ of a Hermitian operator, show that $\sum_n \hat{P}_n = \hat{1}$, where $\hat{1}$ is the unity operator.

4. **Particle in angular momentum eigenstate** (20 points)

A particle is in an eigenstate $|l,m\rangle$ of $L^2$ and $L_z$.

a) (10 points) Show that in this case $\langle L_x \rangle = \langle L_y \rangle = 0$.

b) (10 points) Show that $\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{\hbar^2 l(l+1) - \hbar^2 m^2}{2}$.

Hints: For part a), use $L_+$ and $L_-$. For part b), use $L^2 = L_x^2 + L_y^2 + L_z^2$. 