Sums of Random Variables

Consider \( n \) RVs \( x_i \) and let \( s \equiv \sum_{i=1}^{n} x_i \).

If the RVs are statistically independent, then

\[
< s > = \sum_{i} < x_i >
\]

\[
\text{Var}(s) = \sum_{i} \text{Var}(x_i)
\]
• The individual $p(x_i)$ could be quite different

• Both continuous and discrete RVs could be present

• True for any $n$

• Even if one RV dominates the sum
Results have a special meaning when

1) The means are finite (≠ 0)
2) The variances are finite (≠ ∞)
3) No subset dominates the sum
4) \( n \) is large
Given \( p(x, y) \), find \( p(s \equiv x + y) \)

\[ A \]

\[ p_x, y(x, y) = \alpha - x \]

\[ B \]

\[ P_s(\alpha) = \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\alpha - \zeta} d\eta \ p_{x,y}(\zeta, \eta) \]
\[
\mathbf{C} \quad p_s(\alpha) = \int_{-\infty}^{\infty} d\zeta \ p_{x,y}(\zeta, \alpha - \zeta)
\]

This is a general result; \(x\) and \(y\) need not be S.I.

Application to the Jointly Gaussian RVs in Section 2 shows that \(p(s)\) is a Gaussian with zero mean and a Variance \(= 2\sigma^2(1 + \rho)\).
In the special case that $x$ and $y$ are S.I.

$$p_s(\alpha) = \int_{-\infty}^{\infty} d\zeta \ p_x(\zeta) \ p_y(\alpha - \zeta) = \int_{-\infty}^{\infty} d\zeta' \ p_x(\alpha - \zeta') \ p_y(\zeta')$$

The mathematical operation is called “convolution”.

$$p \otimes q \equiv \int_{-\infty}^{\infty} p(z) q(x - z) \, dz = f(x).$$
Example

Given:

\[ p(z) = \frac{1}{n!a} (z/a)^n \exp(-z/a) \]

\[ q(z) = \frac{1}{m!a} (z/a)^m \exp(-z/a) \]

\[ 0 < z \text{ and } n, m = 0, 1, 2, \cdots \]

Find: \( p \otimes q \)
\[ q(x-z) = q(-(z-x)) \]
\[ p \otimes q = \frac{1}{n!m!} \frac{1}{a^2} \int_0^x \left( \frac{z}{a} \right)^n \left( \frac{x-z}{a} \right)^m e^{-z/a} e^{-(x-z)/a} \, dz \]

\[ = \frac{1}{n!m!} \frac{1}{a} \left( \frac{1}{a} \right)^{n+m+1} e^{-x/a} \int_0^x z^n (x-z)^m \, dz \]

\[ = \frac{1}{n!m!} \frac{1}{a} \left( \frac{x}{a} \right)^{n+m+1} e^{-x/a} \int_0^1 \zeta^n (1-\zeta)^m \, d\zeta \]

\[ = \frac{n!m!}{(n+m+1)!} \frac{1}{(n+m+1)}! \]

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\[ p \otimes q = \frac{1}{(n + m + 1)!} \frac{1}{a} \left( \frac{x}{a} \right)^{n+m+1} e^{-x/a} \]

a function of the same class
Example  Atomic Hydrogen Maser

\[ \nu = 1.4 \ldots \text{GHz} \]

\[ \nu_1 - \nu_0 \text{ about 10 KHz} \]

\[ p(t_{\text{wall}} | n \text{ stays}) = ? \]
\[ t_{\text{wall}} = \sum_{i=1}^{n} t^n, \quad \text{Each stay is S.I.} \]

\[ p(t \mid 1) = \left( \frac{1}{\tau} \right) e^{-t/\tau} \]

\[ p(t \mid 2) = p(t \mid 1) \otimes p(t \mid 1) = \left( \frac{1}{\tau} \right) t^{-\tau} e^{-t/\tau} \]

\[ p(t \mid 3) = p(t \mid 2) \otimes p(t \mid 1) = \left( \frac{1}{2} \right) \left( \frac{1}{\tau} \right) t^{-\tau} e^{-t/\tau} \]
\[ p(t \mid n) = \frac{1}{(n - 1)!} \frac{1}{\tau} \left( \frac{t}{\tau} \right)^{n-1} e^{-t/\tau} \]