Relativistic Velocity Transformation

- Particle velocity \( \vec{u} \) \((u_x, u_y)\)
  in \(x, y, z, t\) system
- What is \( \vec{u}' \) in \(x', y', z', t'\)?
- \(S'\) velocity is \(v\) w.r.t \(S\).

\[
\begin{align*}
  u_x &= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \\
  u_y &= \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}
\end{align*}
\]

Also
\[
\begin{align*}
  u'_x &= \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'} \\
  u'_y &= \lim_{\Delta t' \to 0} \frac{\Delta y'}{\Delta t'}
\end{align*}
\]

Lorentz transformation differentials:

\[
\begin{align*}
  \Delta x' &= \gamma (\Delta x - vt) \\
  \Delta y' &= \Delta y \\
  \Delta t' &= \gamma (\Delta t - \frac{v}{c^2} \Delta x)
\end{align*}
\]

\[
\begin{align*}
  \frac{\Delta x'}{\Delta t'} &= \frac{\gamma (\Delta x - vt)}{\gamma (\Delta t - \frac{v}{c^2} \Delta x)} = \frac{\Delta x / \Delta t - v}{1 - \frac{v}{c^2} (\Delta x / \Delta t)}
\end{align*}
\]

Let \(\Delta t \to 0\)

\[
\begin{align*}
  u'_x &= \frac{u_x - v}{1 - \frac{v}{c^2} u_x}
\end{align*}
\]
Also \[ u'_y = \frac{u_y}{\gamma \left[ 1 - \frac{\nu u_x}{c^2} \right]} \]
\[ u'_z = \frac{u_z}{\gamma \left[ 1 - \frac{\nu u_x}{c^2} \right]} \]

Invert to get
\[ u_x = \frac{u'_x + \nu}{1 + \nu u'_x \frac{c^2}{c^2}} \]
\[ u'_y = \frac{u'_y}{\gamma \left[ 1 + \nu \frac{u'_x}{c^2} \right]} \]
\[ u'_z = \frac{u'_z}{\gamma \left[ 1 + \nu \frac{u'_x}{c^2} \right]} \]

If \( \nu \ll c \)
\[ \begin{align*}
    u'_x &= u_x - \nu \\
    u'_y &= u_y \\
    u'_z &= u_z \\
\end{align*} \]
\(
\text{Galilean Transform.}
\)
Example:

\[ u_x' = \frac{u_x - v}{1 - \frac{v}{c^2}} = \frac{0.9c - (-0.9c)}{1 - \left[(-0.9c)(0.9c)\right]} = \frac{1.8c}{1.81} \]

\[ u_x' = 0.99c \]

Example:

Let \( u_x = c \)

\[ u_x' = \frac{c - v}{1 - \frac{vc}{c^2}} \equiv c \]

Independent of \( v \)!!

Limiting velocity \( \equiv c \)
Doppler Effect

- Sound pitch increases when source approaches.
- Pitch decreases when source recedes.
- What about light?

- Source produces light flashes with period $T_o = 1/\nu_o$ in its rest frame $S$.
- Source moving with velocity $v$ relative to $S$.
- Due to time dilation the period in the observers' rest frame is

$$T = \gamma T_o$$

- Pulse arrives at observer with speed $c$.

Observed frequency is

$$\nu_D = c/L$$ where $L$ is the separation between two pulses. Since source is moving to observer

$$L = c \gamma - v \gamma = (c - v) \gamma$$

and

$$\nu_D = \frac{c}{(c - v) \gamma}$$
\[ \bar{v}_d = \frac{1}{1 - \frac{v}{c}} \quad \frac{1}{\bar{v}_0} \]

\[ \bar{v}_d = \bar{v}_0 \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}} \]

\[ \bar{v}_d = \bar{v}_0 \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \quad \text{(source approaching)} \]

\[ \bar{v}_d = \bar{v}_0 \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \quad \text{(source receding)} \]

\( \bar{v}_d \): frequency in observer's rest frame

\( \bar{v}_0 \): frequency in source rest frame
Doppler Effect: Off-line

Consider source of light making an angle $\theta$ to direction of motion.

Period of flashes,
$\tau = \lambda \tau_0$

Observed frequency,
$\tilde{\nu} = \frac{c}{\lambda}$

$L$: distance between flashes
$\nu_0$: distance source moves between flashes.

$L = c \tau - \nu \tau \cos \theta$
$= \left(c - \nu \cos \theta\right) \tau$

$\tilde{\nu}_0 = \frac{c}{L} = \frac{c}{\left(c - \nu \cos \theta\right) \tau_0 \lambda}$

$\bar{\nu}_0 = \tilde{\nu}_0 \frac{\sqrt{1 - \frac{\nu^2}{c^2}}}{1 - \frac{\nu}{c} \cos \theta}$

$\bar{\nu}_0 (\theta = 90^\circ) = \tilde{\nu}_0 / \lambda$

$\bar{\nu}_0$ (classically $\lambda$) $\equiv 0.$
FIGURE 25.43 Longitudinal and transverse Doppler effects.

FIGURE 2.12 The results of Kindig [9] on the transverse Doppler effect. The experimental points agree very well with the relativistic prediction and not at all with the classical prediction.
Two Observers / Doppler Shift

- Light emitted with rest-frame frequency $\nu_0$.
- $S'$ moves away with speed $V_1$ along x-axis.
- Observer $O_1$ measures redshifted frequency $\nu_1$. 
- $S''$ moves away with speed $V_2$ along x-axis relative to $S'$ and measures a frequency $\nu_2$.
- Assume $S''$ moves with velocity $V$ relative to rest-frame. He should observe $\nu_2$ as frequency shift relative to $\nu_0$.

\[
\begin{align*}
\nu_1 &= \nu_0 \sqrt{\frac{1-V_1/c}{1+V_1/c}} \quad \text{[Observer $S'$]} \\
\nu_2 &= \nu_1 \sqrt{\frac{1-V_2/c}{1+V_2/c}} \quad \text{[Observer $S''$]} \\
\nu_2 &= \nu_0 \sqrt{\frac{1-V_1/c}{1+V_1/c} \times \frac{1-V_2/c}{1+V_2/c}}
\end{align*}
\]

We must also have,
\[
\nu_2 = \nu_0 \sqrt{\frac{1-V_1c}{1+V_1c}}
\]
\[
\frac{1-V/c}{1+V/c} = \frac{[1-v_1/c][1-v_2/c]}{[1+v_1/c][1+v_2/c]}
\]

Solve for \( V \)

\[
V = \frac{v_1+v_2}{1+v_1v_2/c^2}
\]
Twin Paradox

A: Bob
B: Dave

- Twins; one stays and one travels
- A sees B travel a distance $L$ with velocity $v$:
  \[ T = \frac{L}{v} \]
- B reverses direction and returns in same time $T$.
- Assume turn-around can be neglected.
- A observes time $T_B'$ on B's clock is related to his time $T_A = 2T$.

\[ T_B' = \frac{T_A}{\gamma} \]

\[
\frac{\text{Aging of B}}{\text{Aging of A}} = \frac{T_B'}{T_A} = \frac{1}{\gamma} \quad (\text{As viewed by A})
\]

- B's Perspective:
- A travels with velocity $-v$; turns around and comes back.

\[ T_B = 2T \]
\[ T_A' = T_B / \gamma \]

\[
\frac{\text{Aging of B}}{\text{Aging of A}} = \frac{T_B}{T_A'} = \gamma \quad (\text{As viewed by B})
\]
Paradox: A thinks B is younger and B thinks A is younger.

- This problem elicited many papers and discussions for many years. Is something wrong with Special Relativity?

No!!

Resolution:
- Example: Bob leaves his brother in NYC.
- Travels with \( v = 0.8c \) to a star, \( s = 5/3 \).
- By his clock the travel time:
  \[ \bar{t}_B = 3 \text{ yrs} \]

- Fais his rockets and reverses direction and speed.
- At end fires rockets to stop in NYC.

- Total travel time: \( \bar{t}_B = 6 \text{ yrs} \)
- We will ignore the accelerations. Time accelerating can be made much shorter than travel time.
- Dave's clock reads 10 yrs.
- Both twins have identical clocks which send out light signals at one-year intervals.
- Dave gets signals from Bob's clock and records them against the annual signals
of his clock.
- Bob receives the signals from Dave's clock and records them against the signals from his own clock.

![Figure B-2. Spacetime diagram of the twin paradox.](image)

- Dave's world-line is the ct-axis. He stays at x = 0.
- Mark off 10 yrs on the ct-axis.
- Bob's world line is inclined, ct', for a frame moving with v = 0.8c. He stays at x' = 0.
- After 3 yrs he reverses velocity to v = -0.8c and returns.
Mark off 3 yrs going and 3 yrs returning.
Draw light signals from Dave to Bob.
Draw light signals from Bob to Dave.
Signals parallel to light line.

Bob sends 6 signals; the last on arrival.
Dave sends 10 signals; the last on arrival.

How to explain space-time diagram?
Doppler effect.
As clocks recede, their frequency is

\[ f = f_0 \sqrt{\frac{1 - \beta}{1 + \beta}} = \frac{f_0}{3} \]

Bob receives first signal from Dave after 3 yrs, just as he turns around.
Dave receives signals at a reduced rate from Bob on the way out, once every 3 of his years. Receives three signals in nine years.

As clocks approach, their frequencies are

\[ f = f_0 \sqrt{\frac{1 + \beta}{1 - \beta}} = 3f_0 \]

Bob receives 9 signals from Dave on his return journey. Altogether he gets 10 signals.
• Dave receives 3 signals from Bob on the last year. Dave receives 6 signals.

• No disagreement on signals: Bob sends 6; Dave receives 6. Dave sends 10; Bob receives 10.

• Why are the times different?
  • Dave sees Bob recede for 9 years and return for one year.
  • Bob sees himself receding for 3 years and approaching for three years.

• Dave received signals at a slow rate for nine years and a fast rate for one year.
  • Bob had slow signals for three years and fast signals for 3 years.

• Doppler effect shows the asymmetry in the travel.
  • Bob is younger than Dave after trip.

<table>
<thead>
<tr>
<th>Time, y</th>
<th>p (Received)</th>
<th>Turnaround</th>
<th>Q</th>
<th>Bob</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td>11</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>11</td>
<td>11</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>15</td>
<td>14</td>
<td>16</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>16</td>
<td>17</td>
<td>18</td>
<td>18</td>
<td>19</td>
</tr>
<tr>
<td>18</td>
<td>19</td>
<td>20</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time, y</th>
<th>p (Received)</th>
<th>Turnaround</th>
<th>Q</th>
<th>Dave</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td>11</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>11</td>
<td>11</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>15</td>
<td>14</td>
<td>16</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>16</td>
<td>17</td>
<td>18</td>
<td>18</td>
<td>19</td>
</tr>
<tr>
<td>18</td>
<td>19</td>
<td>20</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

Figure B-3. The signal logs for the twins.
Appearance of Moving Objects

- Consider a board of length $L_0$ and width $W_0$ in its rest frame.
- Moving with speed $v$ parallel to $L_0$.
- Take a picture in a plane $\perp v$.
- Picture: Instant simultaneous collection of light from object.

- Consider three points: $A_0, B_0, C_0$.
- Light from $B_0, C_0$ arrives at film plane at same time.
- Light from $A_0$ must start earlier to reach film on time.
  \[ \Delta t = \frac{W_0}{c} \]

- In time $\Delta t$, board moves to the right.
  \[ \Delta x = v \Delta t = \frac{v W_0}{c} \]

Fig. 5-11 (a) A rectangular object moving at speed $v$ parallel to $x$. (b) The apparent positions of the corners $A, B, C$ as recorded at a given instant by a distant observer looking in the $y$ direction in the plane of the object. (c) It is inferred that the object is rotated in its own plane, but not Lorentz-contrasted.

Looking at moving clocks and other objects
At \( \Delta t \):
\[
B_0 \rightarrow B_1 \\
C_0 \rightarrow C_1
\]

- Board along \(-x\) is Lorentz contracted

\( \therefore B_1C_1 = L_0/x \)

Also
\( A'B' = \nu W_0/c \)

Consider board at rest but rotated through angle \( \theta \)

\[
A'B' = W_0 \sin \theta \\
B_1C_1 = B'C' = L_0 \cos \theta
\]

Appears as if board is rotated

\[
\sin \theta = \nu/c \\
\cos \theta = \sqrt{1-\frac{\nu^2}{c^2}}
\]