Relativistic Velocity Transformations

\[ \begin{align*}
\text{Rindler velocity } \mathbf{u'}(u_x', u_y') \\
\text{What is } \mathbf{u'}(u_x', u_y')? \\
\text{S' has velocity } \mathbf{v} \text{ rel to } S \\
\begin{align*}
\mathbf{u}_x' &= \lim_{\Delta t \to 0} \frac{\Delta x'}{\Delta t} \\
\mathbf{u}_y' &= \lim_{\Delta t \to 0} \frac{\Delta y'}{\Delta t} \\
\text{Lorentz-Transform differentials:} \\
\Delta x' &= \gamma (\Delta x - \mathbf{v} \cdot \Delta t) \\
\Delta y' &= \Delta y \\
\Delta t' &= \gamma (\Delta t - \frac{\mathbf{v} \cdot \Delta t}{c^2}) \\
\end{align*}
\end{align*} \]

\[ \begin{align*}
\mathbf{u}_x' &= \frac{u_x - v}{\sqrt{1 - \frac{v^2}{c^2}}} u_x \\
\mathbf{u}_y' &= u_y \\
\mathbf{u}_t' &= u_t \\
\text{``Galilean Transform''} \\
\end{align*} \]

\[ \begin{align*}
\text{Invert to get:} \\
\mathbf{u}_x &= \frac{u_x' + \gamma v u_x}{1 + \frac{\gamma v}{c^2} u_x} \\
\mathbf{u}_y &= \frac{u_y'}{\gamma} \\
\mathbf{u}_t &= \frac{u_t'}{\gamma} \\
\text{Example:} \\
\begin{align*}
\mathbf{u}_x &= 0.9c \\
\mathbf{u}_y &= 0 \\
\mathbf{u}_t &= 0 \\
\end{align*}
\end{align*} \]
**Lecture 35 Blackboard #2**

---

**Example:**
\[
\begin{align*}
  u_x' &= \frac{u_x - v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{0.9c - 0.9c}{\sqrt{1 - (0.9c)^2}} = \frac{1.8c}{1.81} = 0.99c \\
  u_x &= \frac{u_x - v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{u_x - u_x'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{0.9c - 0.9c}{\sqrt{1 - (0.9c)^2}} = \frac{1.8c}{1.81} = 0.99c \\
  v &= \frac{u_x - u_x'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{0.9c - 0.9c}{\sqrt{1 - (0.9c)^2}} = \frac{1.8c}{1.81} = 0.99c
\end{align*}
\]

---

**Doppler Effect (Longitudinal):**
- Sound pitch increases for approaching source.
- Sound pitch decreases for receding source.

**What about light?**
- Source produces flashes with period \( T_0 = \frac{1}{f_0} \) in rest frame \( S' \).
- Source is moving with velocity \( v \) rel. to \( S \).

**Time dilation:**
\[
T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

---

**Pulse travel with speed \( c \):**
- Observed frequency in \( S \) \( f = \frac{c}{L} \).
- Observed frequency in \( S' \) \( f' = \frac{c}{L} \).

**Source is moving with \( v \):**
- Distance between two pulses \( L = cT - vT = (c - v)T \).

---

**Frame \( S \):**
- Frequency in observer's rest frame \( f_0 \).
- Frequency in source's rest frame \( f_0' \).

**Frame \( S' \):**
- Frequency in frame \( S \) \( f \).
- Frequency in frame \( S' \) \( f' \).

---

**Example:**
- \( \text{Doppler effect for light} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}} \) (source approaching).
- \( \text{Doppler effect for light} = \frac{c}{\sqrt{1 + \frac{v^2}{c^2}}} \) (source receding).
Lecture 35 Blackboard #3

Toppler Effect: On-line (Transverse)

Period of flashes $T = \frac{\lambda}{v}$

Observed freq $f_d = \frac{c}{L}$

$L$: dist. between flashes
$v$: dist. source moves bct.
$c$: speed of light

$T \approx \frac{\lambda}{v}$

$\frac{c}{L \pm c \cos \theta} \approx \frac{c}{L}$

$D_d = \frac{c}{L}$

$D_d = \frac{c}{L \sqrt{1 - \frac{v^2}{c^2}}}$

$D_d = \frac{c}{L \sqrt{1 - \frac{v^2}{c^2}}}$

$D_d (\text{Classically}) \approx 0$

Partly Relativistic Effect.

Example: Two Observers - Toppler Shift

Rest frame frequency $D_0$

$S'$ more with $V_1$ (rel to source) measures $D_1$

$S''$ more with $V_2$ (rel to $S'$) measures $D_2$

Must also have:

$D_2 = D_0 \left[ \frac{1 - \frac{V_2}{c}}{1 + \frac{V_2}{c}} \right]$  (Observer $S''$)

$\frac{D_2}{D_0} = \frac{1}{\sqrt{1 - \frac{V_2}{c^2}}}$  (Observer $S''$)

Source At rest

Frequency $D_0$

$D_1 = D_0 \left[ \frac{1 - \frac{V_1}{c}}{1 + \frac{V_1}{c}} \right]$  (Observer $S'$)

$D_2 = D_0 \left[ \frac{1 - \frac{V_2}{c}}{1 + \frac{V_2}{c}} \right]$  (Observer $S''$)

Total change Eq.

$\frac{1 - \frac{V_2}{c}}{1 - \frac{V_1}{c}}$  (Observer $S''$)

Val. Add Eq.'s

$\lambda = \frac{v}{f}$

Period of signal and received signals.
Example: Two Observers - Doppler Shift

Red frame frequency: \( \nu_0 \)

S' moves with \( v_1 \) (relative to source) measures \( \nu_2 \)

S'' moves with \( v_2 \) (relative to S') measures \( \nu_2' \)

Assume also S'' moves with \( V \) (relative to source).

\[ \nu_2 = \frac{\nu_0}{\sqrt{1 - \frac{v_1^2}{c^2}}} \]

\[ \nu_2' = \frac{\nu_0}{\sqrt{1 - \frac{V^2}{c^2}}} \]

Must also have:

\[ \nu_2 = \nu_2' \sqrt{1 - \frac{v_1^2}{c^2}} \]

\[ \nu_2' = \nu_2 \sqrt{1 - \frac{V^2}{c^2}} \]

\[ \frac{V}{c} = \sqrt{1 - \frac{v_1^2}{c^2}} \]

\[ V = \frac{\nu_0}{\nu_0} \frac{\sqrt{1 - \frac{v_1^2}{c^2}}}{\sqrt{1 - \frac{V^2}{c^2}}} \]

Twin Paradox

A: Bob

B: Dave

A sees B travel distance \( L \) with velocity \( V \)

\[ T = \frac{L}{V} \]

B travels and returns in time \( T \).

A observes time \( T_B \) on B's clock.

\[ T_B = \frac{T}{x} \]

A: B is older:

\[ \text{Age}_A - \text{Age}_B = \frac{T_b}{T} = \frac{1}{2} \] (seen by A)

B: A is younger:

\[ \text{Age}_B - \text{Age}_A = \frac{T_B}{T} = \frac{1}{2} \]

Resolution

Bob travels to a star:

\[ v = 0.8c \]

\[ x = 5/3 \]

Travel Time \( T_B = 3 \) yr.

Returns: Total \( T_B = 6 \) yr.

Dave's clock reads \( \frac{T_B}{x} = 10 \) yr.

Both A & B send out signals.

Both light over a year.

Recall sent and received signals.
Lecture 3

Blackboard #5

D: World line: \(ct = \text{axes}, x = 0\).
Mark 10 yrs on \(ct\) - axes.

B: Inclined \(ct\): \(v = 0.8c\).
Stay at \(x = 0\).
Mark 1 yr 3 yrs and turn back.
Light signals // to light line.

D: Sends 6 signals; lost or arrived.
B: Sends 10 signals; last or arrived.

Outbound: Clocks receive:
\[ T = T_0 \sqrt{1 - \frac{v^2}{c^2}} = \frac{T_0}{3} \]
B receives 9 signals in return.
Total: 10!

Inbound: Clocks approach:
\[ T = T_0 \sqrt{1 + \frac{v^2}{c^2}} = 3T_0 \]
D receives 3 signals in last yr.
Total: 6.

D: Sends 6; Dave receives 6.
D: Sends 10; Bob receives 10.

D: Sees 8 receive for 9 yrs and approach for 1 yr.
B: Sees himself receive 3 yrs and approach 3 yrs.

B: Slows 3 yrs.
Fast rate: 1 yr.

1. Apparent Effect - Slower.
Bob is incorrect.

---

Appearance of Moving Objects

Board: length \(L_0\), bulk frame width \(W_0\), moving velocity \(V\), moving velocity \(V\).

Pretend instantaneous collection of light from all points.

Points: \(A, B, C\).

Board: \(\Delta x = \Delta t = \frac{W_0}{c}\).

\[ B_{\alpha} = c + V \frac{W_0}{c} \]

Conclude board at rest by rotating by \(\theta\):

\[ \theta = \frac{W_0}{v} \frac{\sin\theta}{c} \]

\[ \cos\theta = \frac{1 - \frac{v^2}{c^2}}{c^2} = \frac{1}{4} \]

Approximately, board is rotated:

\[ \sin\theta = \frac{1}{2} \frac{\Delta x}{L_0} \frac{v}{c} \]

\[ \cos\theta = \frac{1}{4} \]

\[ \Delta x = \frac{L_0}{4} \]