Boundary Value Problems
Finite Difference Methods

Boundary Conditions with Derivatives
\[ y'' - y' = g(x) \]
\[ y(a) = 0 \]
\[ y'(b) = 0 \]

Central Difference
\[
y_n - 2y_{n+1} + y_{n-1} - h^2y_{n} = h^2g(x_n), \quad n = 1, \ldots, N-1
\]

Difference Equations
\[ y_0 = 0 \]
\[ y_{n-1} - 2y_n + y_{n+1} - h^2y_nx_n = h^2g(x_n), \quad n = 1, \ldots, N-1 \]
\[ y_N = ? \]

Backward Difference
\[ y'(b) = 0 = \frac{y_N - y_{N-1}}{h} + O(h) \]
\[ y_0 = 0 \]
\[ y_{n-1} - 2y_n + y_{n+1} - h^2y_nx_n = h^2g(x_n), \quad n = 1, \ldots, N-1 \]
\[ 2(y_{N-1} - y_N) - h^2y_Nx_N = 0 \quad O(h^3) \]

General Boundary Conditions
\[ p_0y(b) + p_1y'(b) = p_2 \]
\[ p_0y_N + \frac{p_1(y_{N+1} - y_{N-1})}{2h} = p_2 \]

Finite Difference Representation
Add extra point - N equations, N unknowns

2.29 Numerical Marine Hydrodynamics Lecture 17
Numerical Marine Hydrodynamics

- Partial Differential Equations
  - PDE Classification
  - Hyperbolic PDEs
    - Finite Difference Solutions
    - Wave Equation
      - D’Alambert’s Principle
      - Method of Characteristics
  - Parabolic PDEs
  - Elliptical PDEs
Partial Differential Equations

Quasi-linear PDE

\[ A \phi_{xx} + B \phi_{xy} + C \phi_{yy} = F(x, y, \phi, \phi_x, \phi_y) \]

A, B and C Constants

\[
\begin{align*}
B^2 - 4AC &> 0 & \text{Hyperbolic} \\
B^2 - 4AC &= 0 & \text{Parabolic} \\
B^2 - 4AC &< 0 & \text{Elliptic}
\end{align*}
\]
Waves on a String

\[ \rho u^{tt}(x, t) = T u_{xx}(x, t), \quad 0 < x < L, \quad 0 < t < \infty \]

Initial Conditions

\[ u(x, 0) = f(x), \quad 0 \leq x \leq L \]
\[ u_t(x, 0) = g(x), \quad 0 < x < L \]

Boundary Conditions

\[ u(0, t) = 0, \quad 0 < t < \infty \]
\[ u(L, t) = 0, \quad 0 < t < \infty \]

Wave Solutions

\[ u = \begin{cases} 
\quad F(x - ct) & \text{Forward propagating wave} \\
\quad G(x + ct) & \text{Backward propagating wave}
\end{cases} \]

Typically Initial Value Problems in Time, Boundary Value Problems in Space
Time-Marching Solutions – Explicit Schemes Generally Stable
Partial Differential Equations
Parabolic PDE

Heat Flow Equation

\[ \kappa u_{xx}(x, t) = \sigma \rho u_t(x, t), \quad 0 < x < L, \quad 0 < t < \infty \]

Initial Condition

\[ u(x, 0) = f(x), \quad 0 \leq x \leq L \]

Boundary Conditions

\[ u(0, t) = c_1, \quad 0 < t < \infty \]
\[ u(L, t) = c_2, \quad 0 < t < \infty \]

\[ \kappa \text{ Thermal conductivity} \]
\[ \sigma \text{ Specific heat} \]
\[ \rho \text{ Density} \]
\[ u \text{ Temperature} \]
Potential Flow in a Duct
Laplace Equation

\[ u_{xx} + u_{yy} = -v_0 \]

Boundary Conditions
\[
\begin{align*}
    u(x,0) &= f_1(x) \\
    u(x,1) &= f_2(x) \\
    u(0,y) &= f_3(y) \\
    u(1,y) &= f_4(y)
\end{align*}
\]
Partial Differential Equations
Hyperbolic PDE

Wave Equation

\[ \rho u^{tt}(x, t) = T u_{xx}(x, t), \quad 0 < x < L, \quad 0 < t < \infty \]

Discretization

\[ h = \frac{L}{n} \]
\[ k = \frac{T}{m} \]
\[ x_i = (i - 1)h, \quad i = 2, \ldots, n - 1 \]
\[ t_j = (j - 1)k, \quad j = 1, \ldots, m \]

Finite Difference Representations

\[ u_{tt}(x, t) = \frac{u(x_i, t_{j-1}) - 2u(x_i, t_j) + u(x_i, t_{j+1})}{k^2} + O(k^2) \]
\[ u_{xx}(x, t) = \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2} + O(h^2) \]
\[ u_{i,j} = u(x_i, t_j) \]

Finite Difference Representations

\[ \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = \frac{c^2 u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \]
Dimensionless Wave Speed

\[ C = \frac{ck}{h} \]

\[ u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \]

Explicit Finite Difference Scheme

\[ u_{i,j+1} = (2 - 2C^2)u_{i,j} + C^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}, \quad i = 2, \ldots, n - 1 \]

Stability Requirement

\[ C = \frac{ck}{h} < 1 \]
Partial Differential Equations

Hyperbolic PDE

Euler Starter

\[ u_{i,2} = u(x_i, k) \simeq u(x_i, 0) + ku_i(x_i, 0)k = f(x_i) + kg(x_i) \]

Second Derivative Known

\[ u_{xx}(x_i, 0) = f'' \]

From Wave Equation

\[ u_{tt}(x_i, 0) = c^2 u_{xx}(x_i, 0) = c^2 f_{i-1} - 2f_i + f_{i+1} + O(h^2) \]

Taylor Expansion

\[ u(x, k) = u(x, 0) + u_t(x, 0) + \frac{u_{tt}(x, 0)k^2}{2} + O(k^3) \]

Higher Order Self Starter

\[ u_{i,2} = u(x_i, k) = f_i + kg_i + \frac{c^2k^2}{2h^2}(f_{i-1} - f_i + f_{i+1}) + O(h^2k^2) + O(k^3) \]

\[ = (1 - C^2) f_i + kg_i + \frac{C^2}{2}(f_{i+1} + f_{i-1}) \]
Wave Equation

d’Alembert’s Solution

Wave Equation

\[ \rho u^{tt}(x, t) = T u_{xx}(x, t), \quad 0 < x < L, \quad 0 < t < \infty \]

Solution

\[ u(x, t) = F(x - ct) + G(x + ct), \quad 0 < x < L \]

Periodicity

\[ F(-z) = -F(z) \]
\[ f(z + 2L) = F(z) \]
\[ G(-z) = -G(z) \]
\[ G(z + 2L) = G(z) \]

Proof

\[ u_{xx}(x, t) = F''(x - ct) + G''(x + ct) \]
\[ u_{tt}(x, t) = c^2 F''(x - ct) + c^2 G''(x + ct) \]
\[ = c^2 u_{xx}(x, t) \]
Hyperbolic PDE
Method of Characteristics

Explicit Finite Difference Scheme

\[ u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \]

\[ u_{i,j+1} = (2 - 2C^2)u_{i,j} + C^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}, \quad i = 2, \ldots n - 1 \]

First 2 Rows known

\[ u_{i,1} = u(x_i, 0) \]
\[ u_{i,2} = u(x_i, k) \]

Characteristic Sampling

\[ k = h/c \Rightarrow C = 1 \]

Exact Discrete Solution

\[ u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \]
Hyperbolic PDE
Method of Characteristics

Exact Discrete Solution

\[ u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \]

D’Alembert’s Solution

\[ x_i - ct_j = (i - 1)h - c(j - 1)k \]
\[ = (i - 1)h - (j - 1)h \]
\[ = (i - j)h \]

\[ x_i + ct_j = (i - 1)h + c(j - 1)k \]
\[ = (i - 1)h + (j - 1)h \]
\[ = (i + j - 2)h \]

\[ u_{i,j} = F((i - j)h) + G((i + j - 2)h) \]

Proof

\[ u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \]
\[ = F((i + 1 - j)h) + F((i - 1 - j)h) - F((i - (j - 1))h) \]
\[ + G((i + 1 + j - 2)h) + G((i - 1 + j - 2)h) - G((i + j - 1 - 2)h) \]
\[ = F((i - (j + 1))h) + G((i + (j + 1) - 2)h) \]
\[ = u_{i,j+1} \]

Numerical Marine Hydrodynamics

Lecture 17
Waves on a String

\[ L = 10; \]
\[ T = 10; \]
\[ c = 1.5; \]
\[ N = 100; \]
\[ h = L/N; \]
\[ M = 400; \]
\[ k = T/M; \]
\[ C = c*k/h \]
\[ Lf = 0.5; \]
\[ x = [0:h:L]; \]
\[ t = [0:k:T]; \]
\[ \% fx = [exp(-0.5*(\text{num2str}(L/2) - x).^2/(\text{num2str}(Lf)).^2)]; \]
\[ \% gx = '0'; \]
\[ fx = \text{exp}(-0.5*(5-x).^2/0.5^2).*\text{cos}((x-5)*\pi); \]
\[ gx = '0'; \]
\[ f = \text{inline}(fx,'x'); \]
\[ g = \text{inline}(gx,'x'); \]
\[ n = \text{length}(x); \]
\[ m = \text{length}(t); \]
\[ u = \text{zeros}(n,m); \]
\[ u(2:n-1,1) = f(x(2:n-1)); \]
\[ \text{for } i = 2:n-1 \]
\[ u(i,2) = (1-C^2)*u(i,1) + k*gx(x(i)) + C^2*(u(i-1,1)+u(i+1,1))/2; \]
\[ \text{end} \]
\[ \text{for } j = 2:m-1 \]
\[ \text{for } i = 2:n-1 \]
\[ u(i,j+1) = (2-2*C^2)*u(i,j) + C^2*(u(i+1,j)+u(i-1,j)) - u(i,j-1); \]
\[ \text{end} \]
\[ \text{end} \]