APPENDIX A. UNIQUENESS OF LIKELIHOOD RATIOS

Let \((X, \mathcal{B})\) be a measurable space. For two measures \(\mu\) and \(\nu\) on \(\mathcal{B}\), \(\mu\) is said to be absolutely continuous with respect to \(\nu\) if for each \(B \in \mathcal{B}\), \(\nu(B) = 0\) implies \(\mu(B) = 0\). Recall that the measure \(\nu\) is called \(\sigma\)-finite if \(X\) is the union of a sequence of sets \(B_n \in \mathcal{B}\) with \(\nu(B_n) < \infty\). Recall also that by the Radon-Nikodym theorem (e.g. RAP, Theorem 5.5.4), if \(\mu\) is finite (i.e. \(\mu(X) < \infty\)), \(\nu\) is \(\sigma\)-finite, and \(\mu\) is absolutely continuous with respect to \(\nu\), then there is an integrable function \(\frac{d\mu}{d\nu} := f \geq 0\) such that

\[
\int_B f \, d\nu = \mu(B) \quad \text{for all } B \in \mathcal{B}.
\]

Then for any integrable function \(g\) for \(\mu\),

\[
(A.1) \quad \int g f \, d\nu = \int g \, d\mu;
\]

this is true for indicator functions \(f = 1_B\), thus for simple functions, then by monotone convergence for nonnegative measurable functions, then by linearity for any \(\mu\)-integrable function.

Now, if \(\kappa\) is a finite measure absolutely continuous with respect to \(\mu\), then letting \(g := 1_B \frac{d\kappa}{d\mu}\) for any \(B \in \mathcal{B}\), which is \(\mu\)-integrable, we get from (A.1)

\[
\int_B \frac{d\kappa}{d\mu} \frac{d\mu}{d\nu} \, d\nu = \int_B \frac{d\kappa}{d\mu} \, d\mu = \kappa(B).
\]

It follows that a.e. \(\nu\), i.e. almost everywhere for \(\nu\),

\[
(A.2) \quad \frac{d\kappa}{d\nu} = \frac{d\kappa}{d\mu} \frac{d\mu}{d\nu}.
\]

Let \(P\) and \(Q\) be probability measures on \((X, \mathcal{B})\) and for a \(\sigma\)-finite measure \(\nu\) such that \(P\) and \(Q\) are both absolutely continuous with respect to \(\nu\), a form of likelihood ratio of \(Q\) to \(P\) is defined by

\[
R_{Q/P,\nu}(x) := \begin{cases} \frac{dQ}{dP} \frac{dP}{d\nu}(x), & \text{if } (dP/d\nu)(x) > 0 \\ +\infty, & \text{if } (dQ/d\nu)(x) > 0 = (dP/d\nu)(x) \\ 0, & \text{if } (dQ/d\nu)(x) = (dP/d\nu)(x) = 0. \end{cases}
\]

The likelihood ratio actually doesn’t depend on \(\nu\):

A.3 Theorem. If \(P\) and \(Q\) are probability measures on \((X, \mathcal{B})\), both absolutely continuous with respect to \(\sigma\)-finite measures \(\nu\) and \(\tau\) on \((X, \mathcal{B})\), then

\[
R_{Q/P,\nu}(x) = R_{Q/P,\tau}(x)
\]
for \((P + Q)\)-almost all \(x\).

**Proof.** We can assume \(\tau = P + Q\), which is absolutely continuous with respect to \(\nu\). Then by (A.2),

\[
\frac{dP}{d\nu} = \frac{dP}{d\tau} \frac{d\tau}{d\nu} \quad \text{and} \quad \frac{dQ}{d\nu} = \frac{dQ}{d\tau} \frac{d\tau}{d\nu}, \; \nu\text{-almost everywhere.}
\]

Thus

\[
R_{Q/P,\nu} = \frac{dQ/d\nu}{dP/d\nu} = \frac{dQ/d\tau}{dP/d\tau} = R_{Q/P,\tau}
\]

on the set where \(d\tau/d\nu > 0\) and \(dP/d\tau > 0\); \(\nu\)-almost everywhere on this set, \(dP/d\nu > 0\) by (A.4). On the set where \(d\tau/d\nu > 0 = dP/d\tau\), we have almost everywhere for \(\nu\), \(dQ/d\nu > 0 = dP/d\nu\), so \(R_{Q/P,\nu} = +\infty = R_{Q/P,\tau}\). The set where \(d\tau/d\nu = 0\) has \((P + Q)\)-measure 0, so \(R_{Q/P,\nu} = R_{Q/P,\tau}\) almost everywhere for \(P + Q\). \(\square\)

Recall that \(R_{Q/P}\) was defined in Sec. 1.1 as \(R_{Q/P,\tau}\) for \(\tau = P + Q\). Thus Theorem A.3 gives \(R_{Q/P,\nu} = R_{Q/P} \; (P + Q)\)-almost everywhere, for any \(\sigma\)-finite \(\nu\) dominating \(P\) and \(Q\).