Partial solutions to problem set 11


**Problem 129.8a)** \( f(x) = x^3 \) on \((0,1)\). The odd \(2l\)-periodic extension of \( f \) is

![Graph of \( f(x) = x^3 \)]

which is \( L^2 \), piecewise \( C^1 \), but not continuous. So exactly the same arguments as in \#7 give that the Fourier sine series of \( f \) converges to \( f \) in \( L^2 \), to \( f \) pointwise on \([0,l]\), to \( \frac{1}{2}[f(l^+) + f(l^-)] = \frac{1}{2}[0 + 0] = 0 \) at \( x = 0 \), and it does not converge uniformly.

**b)** \( f(x) = l x - x^2 = x(l - x) \). The odd \(2l\)-periodic extension of \( f \) is now \( C^1 \), in fact piecewise \( C^2 \) (2nd deriv = -2 on \((0,1)\), 2 on \((-1,0)\)).

![Graph of \( f(x) = l x - x^2 \)]

This suffices to give uniform convergence of the Fourier series, hence pointwise and \( L^2 \) convergence as well. To use Theorem 2 directly note that \( f \) is \( C^2 \) on \([0,l]\) and satisfies Dirichlet BC’s (the BC’s of the Fourier sine series), so by Theorem 2 the Fourier sine series converges to \( f \) uniformly.

**c)** \( f(x) = x^2 \sin \frac{nx}{l} \) does not even converge, since the integrand is \( \geq x^2 \frac{\pi^2}{l^2} \cdot 1 = \frac{\pi^2}{l^2} \) near \( x = 0 \), \& \( \frac{1}{l} \) is not integrable.

\((\sin \theta = \theta + \) higher order terms in Taylor series, so \( \sin \theta \geq \frac{\theta}{2} \) near 0).

Thus, the Fourier sine series of \( f \) makes no sense, so we cannot talk about its convergence either.

**Problem 129.16:** \( \varphi(x) = |x| \) in \((-\pi, \pi)\). Want to approximate \( \varphi \) by \( f(x) = \frac{1}{2} a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x \) and minimize the \( L^2 \) error \( ||f - \varphi||^2 = \int_0^\pi |f(x) - \varphi(x)|^2 dx \).

Since \( f \) is just a linear combination of the orthogonal functions 1, \( \cos x, \sin x, \cos 2x, \sin 2x \), the minimizing choice is given by the Fourier coefficients (Theorem 5), i.e. if we write

\[
\varphi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx),
\]
then \(a_0 = A_0, a_1 = A_1, a_2 = A_2, b_1 = B_1, b_2 = B_2\).

But \(\varphi\) is even, so the \(B_n\) are zero, hence \(b_1 = b_2 = 0\), and

\[
\begin{align*}
  a_0 &= A_0 = \frac{2}{\pi} \int_0^\pi x \, dx = \frac{2}{\pi} \frac{x^2}{2} \Big|_0^\pi = \pi \\
  a_n &= A_n = \frac{2}{\pi} \int_0^\pi x \cos nx \, dx = \frac{2}{\pi n} x \sin nx \Big|_0^\pi = \frac{2}{n^2 \pi} \cos nx \Big|_0^\pi \\
  &= -\frac{2}{n\pi} \int_0^\pi \sin nx \, dx = -\frac{2}{n\pi} \cos nx \Big|_0^\pi = -\frac{2}{n\pi}((-1)^n - 1), \quad n \geq 1
\end{align*}
\]

So \(a_1 = -\frac{4}{\pi}, a_2 = 0\).