Solutions to Problem Set #1

1-1 pg. 12 #9

\[ B_n = \bigcup_{i=n}^{\infty} A_i, \quad C_n = \bigcap_{i=n}^{\infty} A_i \]

a) \[ B_n \supset B_{n+1} \]

\[ B_n = A_n \cup (\bigcup_{i=n+1}^{\infty} A_i) = A_n \cup B_{n+1} \]

\[ s \in B_{n+1} \Rightarrow s \in B_{n+1} \cup A_n = B_n \]

\[ C_n \subset C_{n+1} \]

\[ C_n = A_n \cap C_{n+1} \]

\[ s \in C_n = A_n \cap C_{n+1} \Rightarrow s \in C_{n+1} \]

b) \[ s \in \bigcap_{i=1}^{\infty} B_i \Rightarrow s \in B_n \] for all \( n \)

\[ s \in \bigcup_{i=1}^{\infty} A_i \] for all \( n \) \( \Rightarrow s \in \) some \( A_i \) for \( i \geq n \), for all \( n \)

\[ \Rightarrow s \in \text{ infinitely many events } A_i \Rightarrow A_i \text{ happen infinitely often.} \]

c) \[ s \in \bigcup_{i=1}^{\infty} C_n \Rightarrow s \in \) some \( C_n = \bigcap_{i=n}^{\infty} A_i \Rightarrow \) for some \( n \), \( s \in \) all \( A_i \) for \( i \geq n \)

\[ \Rightarrow s \in \text{ all events starting at } n. \]

1-2 pg. 18 #4

\[ P(\text{at least 1 fails}) = 1 - P(\text{neither fail}) = 1 - 0.4 = 0.6 \]

1-3 pg. 18 #12

\[ A_1, A_2, \ldots \]

\[ B_1 = A_1, B_2 = A_1^c A_2, \ldots, B_n = A_1^c \ldots A_{n-1}^c A_n \]

\[ P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(B_i) \] splits the union into disjoint events, and covers the entire space.

follows from: \[ \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i \]

take point \( s \) in \[ \bigcup_{i=1}^{n} A_i \Rightarrow s \in \) at least one \( \Rightarrow s \in A_1 = B_1 \),

if not, \( s \in A_1^c \), if \( s \in A_2 \), then \( s \in A_1^c A_2 = B_2 \), if not... etc.

at some point, the point belongs to a set.

The sequence stops when \( s \in A_1^c \cap A_2^c \cap \ldots \cap A_{k-1}^c \cap A_k = B_k \)

\[ \Rightarrow s \in \bigcup_{i=1}^{k} B_i, P(\bigcup_{i=1}^{n} A_i) = P(\bigcup_{i=1}^{n} B_i) \]

\[ = \sum_{i=1}^{n} P(B_i) \text{ if } B_i \text{'s are disjoint.} \]

Should also prove that the point in \( B_i \) belongs in \( A_i \). Need to prove \( B_i \)'s disjoint - by construction:

\[ B_i, B_j \Rightarrow B_i = A_1^c \cap \ldots \cap A_{i-1}^c \cap A_i \]

\[ B_j = A_1^c \cap \ldots \cap A_{j-1}^c \cap A_j \]

\[ s \in B_i \Rightarrow s \in A_i, s' \in B_j \Rightarrow s' \notin A_i. \]

\[ \Rightarrow \text{ implies that } s \neq s' \]

1-4 pg. 27 #5

\[ #(S) = 6 \times 6 \times 6 \times 6 = 6^4 \]

\[ #(\text{different}) = 6 \times 5 \times 4 \times 3 = P_{6,4} \]

\[ P(\text{all different}) = \frac{P_{6,4}}{6!} = \frac{5}{18} \]

1-5 pg. 27 #7

12 balls in 20 boxes.

\[ P(\text{no box receives > 1 ball, each box will have 0 or 1 balls}) \]

also means that all balls fall into different boxes.

\[ #(S) = 20^{12} \]

\[ #(\text{all different}) = 20 \times 19 \ldots \times 9 = P_{20,12} \]
\[ P(...) = \frac{P_{20,12}}{20^{12}} \]

1-6 pg. 27 #10
100 balls, \( r \) red balls.
\( A_i = \{ \text{draw red at step } i \} \)
think of arranging the balls in 100 spots in a row.
a) \( P(A_1) = \frac{r}{100} \)
b) \( P(A_{50}) \)
sample space = sequences of length 50.
\( \#(S) = 100 \times 99 \times \ldots \times 50 = P_{100,50} \)
\( \#(A_{50}) = r \times P_{99,49} \) red on 50. There are 99 balls left, \( r \) choices to put red on 50.
\( P(A_{50}) = \frac{r}{100} \), same as part a.
c) As shown in part b, the particular draw doesn’t matter, probability is the same.
\( P(A_{100}) = \frac{r}{100} \)

1-7 pg. 34 #6
Seat \( n \) people in \( n \) spots.
\( \#(S) = n! \)
\( \#(AB \text{ sit together}) = ? \)
visualize \( n \) seats, you have \( n-1 \) choices for the pair.
2(\( n-1 \)) ways to seat the pair, because you can switch the two people.
but, need to account for the \( (n-2) \) people remaining!
\( \#(AB) = 2(n-1)(n-2)! \)
therefore, \( P = \frac{2(n-1)!}{n!} = \frac{2}{n} \)
or, think of the pair as 1 entity. There are \( (n-1) \) entities, permute them, multiply by 2 to swap the pair.

1-8 pg. 34 #11
Out of 100, choose 12.
\( \#(S) = \binom{100}{12} \)
\( \#(AB \text{ are on committee}) = \binom{98}{10} \), choose 10 from the 98 remaining.
\( P = \frac{\binom{98}{10}}{\binom{100}{12}} \)

1-9 pg. 34 #16
50 states \( \times \) 2 senators each.
a) Select 8 , \( \#(S) = \binom{100}{8} \)
\( \#(\text{state 1 or state 2}) = \binom{98}{2} + \binom{7}{2} \binom{98}{7} \)
or, calculate: \( 1 - P(\text{neither chosen}) = 1 - \frac{\binom{98}{10}}{\binom{100}{12}} \)
b) \( \#(\text{one senator from each state}) = 2^{50} \)
select group of 50 = \( \binom{100}{50} \)

1-10 pg. 34 #17
In the sample space, only consider the positions of the aces in the hands.
\( \#(S) = \binom{52}{4}, \#(\text{all go to 1 player}) = 4 \times \binom{13}{3} \)
\( P = 4 \times \frac{\binom{13}{3}}{\binom{52}{4}} \)

1-11
\( r \) balls, \( n \) boxes, no box is empty.
first of all, put 1 ball in each box from the beginning. 
\( r-n \) balls remain to be distributed in \( n \) boxes.
\[
\binom{n + (r - n) - 1}{r - n} = \binom{r - 1}{r - n}
\]

**1-12**

30 people, 12 months.
\[P(6 \text{ months with 3 birthdays}, 6 \text{ months with 2 birthdays})\]
\[#(S) = 12^{30}\]
Need to choose the 6 months with 3 or 2 birthdays, then the multinomial coefficient:
\[
#(\text{possibilities}) = \binom{12}{6} \binom{30}{3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 2}
\]

**End of Lecture 6**