§2.2 Independence of events.

\[ P(AB) = \frac{P(AB)}{P(B)} \]

Definition - A and B are independent if

\[ P(AB) = P(A)P(B) \]

Experiments can be physically independent (roll 1 die, then roll another die), or seem physically related and still be independent.

Example: A = \{odd\}, B = \{1, 2, 3, 4\}. Related events, but independent.

\[ P(A) = \frac{1}{2}, P(B) = \frac{2}{3}, AB = \{1, 3\} \]

\[ P(AB) = \frac{1}{2} \times \frac{2}{3} = P(AB) = \frac{1}{3}, \text{ therefore independent.} \]

Independence does not imply that the sets do not intersect.

If A, B are independent, find \( P(AB^c) \)

\[ P(AB) = P(A)P(B) \]

\( AB^c = A \setminus AB, \text{ as shown:} \)

so, \( P(AB^c) = P(A) - P(AB) \)

\[ = P(A) - P(A)P(B) \]

\[ = P(A)(1 - P(B)) \]

\[ = P(A)P(B^c) \]

therefore, A and \( B^c \) are independent as well.

similarly, \( A^c \) and \( B^c \) are independent. See Pset 3 for proof.

Independence allows you to find \( P(\text{intersection}) \) through simple multiplication.
Example: Toss an unfair coin twice, these are independent events.
\[ P(H) = p, \ 0 \leq p \leq 1, \ \text{find } P(\text{"TH"}) = \text{tails first, heads second} \]
\[ P(\text{"TH"}) = P(T)P(H) = (1 - p)p \]
Since this is an unfair coin, the probability is **not** just \( \frac{1}{4} \).
If fair, \( \frac{TH}{HH + HT + TH + TT} = \frac{1}{4} \)

If you have several events: \( A_1, A_2, \ldots A_n \) that you need to prove independent:
It is necessary to show that **any** subset is independent.

Total subsets: \( A_{i1}, A_{i2}, \ldots, A_{ik}, 2 \leq k \leq n \)
Prove: \( P(A_{i1}A_{i2}...A_{ik}) = P(A_{i1})P(A_{i2})...P(A_{ik}) \)
You could prove that any 2 events are independent, which is called "pairwise" independence, but this is not sufficient to prove that all events are independent.

Example of pairwise independence:
Consider a tetrahedral die, equally weighted.
Three of the faces are each colored red, blue, and green,
but the last face is multicolored, containing red, blue and green.
\[ P(\text{red}) = \frac{2}{4} = 1/2 = P(\text{blue}) = P(\text{green}) \]
\[ P(\text{red and blue}) = 1/4 = 1/2 \times 1/2 = P(\text{red})P(\text{blue}) \]
Therefore, the pair \{red, blue\} is independent.
The same can be proven for \{red, green\} and \{blue, green\}.
but, what about all three together?
\[ P(\text{red, blue, and green}) = 1/4 \neq P(\text{red})P(\text{blue})P(\text{green}) = 1/8, \text{not fully independent.} \]

Example: \( P(H) = p, P(T) = 1 - p \) for unfair coin
Toss the coin 5 times \( \sim \) \( P(\text{"HTHTT"}) \)
\[ = P(H)P(T)P(H)P(T)P(T) \]
\[ = p(1 - p)p(1 - p)(1 - p) = p^2(1 - p)^3 \]

Example: Find \( P(\text{get 2H and 3T, in any order}) \)
\[ = \text{sum of probabilities for ordering} \]
\[ = P(\text{HHTTT}) + P(\text{HTHTT}) = \ldots \]
\[ = p^2(1 - p)^3 + p^2(1 - p)^3 + \ldots \]
\[ = (\frac{2p}{3})^2(1 - p)^3 \]

General Example: Throw a coin \( n \) times, \( P(\text{k heads out of n throws}) \)
\[ = \binom{n}{k}p^k(1 - p)^{n-k} \]

Example: Toss a coin until the result is "heads;" there are \( n \) tosses before \( H \) results.
\( P(\text{number of tosses = } n) = ? \)
needs to result as "TTT....TH," number of T's = \( (n - 1) \)
\[ P(\text{tosses = } n) = P(\text{TT}...\text{H}) = (1 - p)^{n-1}p \]

Example: In a criminal case, witnesses give a specific description of the couple seen fleeing the scene.
\( P(\text{random couple meets description}) = 8.3 \times 10^{-8} = p \)
We know at the beginning that 1 couple exists. Perhaps a better question to be asked is:
Given a couple exists, what is the probability that another couple fits the same description?
\[ P(2 \text{ couples exists}) \]
\[ A = P(\text{at least 1 couple}), B = P(\text{at least 2 couples}), \text{ find } P(B|A) \]
\[ P(B|A) = \frac{P(B)P(A)}{P(A)} = \frac{P(B)}{P(A)} \]
Out of n couples, \( P(A) = P(\text{at least 1 couple}) = 1 - P(\text{no couples}) = 1 - \prod_{i=1}^{n} (1 - p) \)

*Each* couple doesn’t satisfy the description, if no couples exist.

Use independence property, and multiply.

\[
P(A) = (1 - p)^n
\]

\[
P(B) = P(\text{at least two}) = 1 - P(0 \text{ couples}) - P(\text{exactly 1 couple})
\]

\[
= 1 - (1 - p)^n - n \times p(1 - p)^{n-1}, \text{ keep in mind that } P(\text{exactly 1}) \text{ falls into } P(k \text{ out of } n)
\]

\[
P(B|A) = \frac{1 - (1 - p)^n - np(1 - p)^{n-1}}{1 - (1 - p)^n}
\]

If \( n = 8 \text{ million people}, \ P(B|A) = 0.2966, \text{ which is within reasonable doubt!} \)

\( P(2 \text{ couples}) < P(1 \text{ couple}), \text{ but given that 1 couple exists, the probability that 2 exist is not insignificant.} \)

In the large sample space, the probability that B occurs when we know that A occurred is significant!

**2.3 Bayes’s Theorem**

It is sometimes useful to separate a sample space \( S \) into a set of disjoint partitions:

\[
B_i \cap B_j = \emptyset, \text{ for } i \neq j, S = \bigcup_{i=1}^{k} B_i \text{ (disjoint)}
\]

Total probability:

\[
P(A) = \sum_{i=1}^{k} P(AB_i) = \sum_{i=1}^{k} P(A|B_i)P(B_i)
\]

(all \( AB_i \) are disjoint, \( \bigcup_{i=1}^{k} AB_i = A \))

** End of Lecture 5**