\[ \langle \hat{p} \rangle^* = \langle \hat{p} \rangle^* = \int dx \, \bar{\psi} \left( i \frac{\partial}{\partial x} \right) \psi^* = i \int dx \, \bar{\psi} \left( i \frac{\partial}{\partial x} \right) \psi^* - \int \left( \frac{\partial}{\partial x} \left( \bar{\psi} \psi^* \right) \right) = \int dx \, \bar{\psi} \left( i \frac{\partial}{\partial x} \right) \psi = \langle \hat{p} \rangle, \]

where again we have used integration by parts and the fact that \( \psi \) vanishes at \( \pm \infty \).

Consequently \( \langle \hat{p} \rangle = \langle \hat{p} \rangle^* \), i.e. all expectation values of \( \hat{p} \) are real.

Since an eigenvalue is the expectation value for the corresponding stationary state, all eigenvalues of the momentum operator must be real.

An operator whose eigenvalues are all real (or equivalently, whose expectation value for all admissible wavefunctions is real) is called a **Hermitean operator**.

**Physically measurable quantities are represented by Hermitean operators.**

Similarly, one can show that \( \langle E \rangle^* = \langle E \rangle \) for any state, so all energy eigenvalues are real. The Hamiltonian operator \( \hat{H} \) is a Hermitean operator.
Can we \textit{derive} Newton's \( f = ma \) from the SE?

\[ f = -\frac{dV}{dx} - ma = \frac{df}{dt} \]

Let us calculate the expectation value of \( \frac{df}{dt} \):

\[
\langle \frac{df}{dt} \rangle = \frac{d}{dt} \langle \rho \rangle = \frac{d}{dt} \int dx \Psi^*(x,t) \left( \frac{\delta}{\delta x} \right) \Psi(x,t) = \\
= \frac{1}{i} \int dx \left( \frac{\delta \Psi^*}{\delta x} \frac{dV}{dx} + \Psi^* \frac{\delta}{\delta x} \frac{d\Psi}{dx} \right) = \\
= \int dx \left[ -\frac{V}{2m} \frac{\delta^2 \Psi^*}{\delta x^2} \frac{d\Psi}{dx} + \frac{V}{2m} \frac{\delta \Psi^*}{\delta x} \frac{d^2 \Psi}{dx^2} - \frac{\delta \Psi^*}{\delta x} \left( \frac{\delta \Psi}{\delta x} - V \Psi \right) \right]
\]

The integrand is

\[
A = -\frac{V}{2m} \frac{\delta^2 \Psi^*}{\delta x^2} \frac{d\Psi}{dx} + \frac{V}{2m} \frac{\delta \Psi^*}{\delta x} \frac{d^2 \Psi}{dx^2} - \frac{\delta \Psi^*}{\delta x} \left( \frac{\delta \Psi}{\delta x} - V \Psi \right) = \\
= \frac{V}{2m} \frac{\delta \Psi^*}{\delta x} \left[ \delta \left( \frac{\delta \Psi}{\delta x} \right) - \frac{\delta^2 \Psi^*}{\delta x^2} \left( \frac{\delta \Psi}{\delta x} \right) \right] - \frac{\delta \Psi^*}{\delta x} \left( \frac{\delta \Psi}{\delta x} \right) \\
\]

Again the integral over the first term vanishes since \( \Psi \to 0 \) for \( x \to \pm \infty \), and we are left with

\[
\langle \frac{df}{dt} \rangle = \int dx \Psi^*(x,t) \left( -\frac{\delta V}{\delta x}(x) \right) \Psi(x,t) = \langle -\frac{dV}{dx} \rangle \quad \text{or}
\]

\[
\frac{d}{dt} \langle \rho \rangle = -\langle \frac{dV}{dx} \rangle
\]

It follows from the SE that the expectation values obey the classical equation of motion.
Average momentum changes due to average force
\[-\langle \frac{dV}{dx} \rangle = -\int dx \psi^* \left( \frac{\partial V}{\partial x} \right) \psi = \int dx \ F(x) \ |\phi(x,t)|^2,\]
i.e. position-dependent force \(F(x)\) is weighted by probability density \(|\phi(x,t)|^2\) for finding the particle at position \(x\) at time \(t\).

Note, however, that \(\langle \frac{dV}{dx} \rangle \neq \frac{d}{dx} V(\langle x \rangle)\)

Example 1: Split wave packet

\[
\frac{dV}{dx}\text{ here determines motion}\ (\langle \frac{df}{dt} \rangle)
\]

for this force \(\frac{d}{dt} \langle p \rangle = 0\)
Example 2: Force varying quickly on wavepacket scale

Classical calculation \(-\frac{d}{dx} V(x)\) would predict very large (and quickly varying force as \(x\) changes), actual QM force \(-\frac{dV}{dx}\) experienced by particle is much smaller.

However, if the force varies slowly compared to the size of a single wavepacket, then
\[
\langle -\frac{dV}{dx} \rangle = \langle f(x) \rangle = -\frac{d}{dx} V(x)
\]
This is the reason why we can treat particles in macroscopic potentials usually as classical particles.

\[
\begin{align*}
\frac{dp}{dt} & = -\langle \frac{dV}{dx} \rangle \\
\frac{dp}{dt} & = -\frac{d}{dx} V(x)
\end{align*}
\]
always true

Ehrenfest's theorem for slowly varying potentials
Eigenfunctions of the momentum operator

Let us find the eigenfunctions \( u_p \) of the momentum operator, i.e. the eigenfunctions satisfying

\[
\hat{p} u_p = p u_p
\]

where \( p \) is some (fixed) particular eigenvalue of \( \hat{p} \).

We know that \( \hat{p} \) is Hermitian, so all eigenvalues \( p \) are real.

In position space, we have \( \hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x} \) and

\[
\frac{\hbar}{i} \frac{\partial}{\partial x} u_p(x) = p u_p(x)
\]

\[
\Rightarrow u_p(x) = A_p e^{ipx/\hbar}
\]

The momentum eigenfunctions are (of course) just the plane waves.

Let us check the orthogonality condition for eigenstates:

\[
\int \! dx \, u_p^* (x) u_{p'} (x) = A_p^* A_{p'} \int \! dx \, e^{-ip'x/\hbar} e^{ipx/\hbar}
\]

\[
= A_p^* A_{p'} \int \! dx \, e^{i(p-p')x/\hbar} = t A_p^* A_{p'} \int \! dy \, e^{i(p-p')y}
\]

\[
= t A_p^* A_{p'} \delta(p-p') 2\pi = t 2\pi A_p^* A_{p'} \delta(p-p')
\]

The momentum eigenfunctions are orthogonal for \( p \neq p' \), but we have a normalization problem for \( p = p' \): The
Dirac delta function diverges, or equivalently, the integral
\[
\int dx \left| u_\alpha(x) \right|^2 = |A_\alpha|^2 \int dx \left| e^{i}px/k \right|^2 = 1
\]
diverges.

Before looking at possibilities to deal with the normalization problem, let us calculate the expansion coefficients \( c(\rho) \)
\[
\psi(x) = \int d\rho \; c(\rho) \; u_\rho(x) \quad \text{with} \quad c(\rho) = \int dx \; u_\rho^*(x) \; \psi(x)
\]
Then
\[
c(\rho) = \int dx \; A_\rho^* e^{-i}px/k \psi(x) = A_\rho^* \int dx \; \psi(x) e^{-i}px/k
\]
\[
= \sqrt{2\pi k} \; \phi(\rho)
\]
We see that if we make the normalization choice
\[
A_\rho = \frac{1}{\sqrt{2\pi k}}
\]
then the expansion coefficients \( c(\rho) \)
into momentum eigenstates are just given by the Fourier transform
\[
\psi(x) = \frac{1}{\sqrt{2\pi k}} \; e^{i}px/k \quad \text{"normalised" momentum eigenstates}
\]
\[
\psi(x) = \int d\rho \; \phi(\rho) \; u_\rho(x) = \frac{1}{\sqrt{2\pi k}} \int d\rho \; \phi(\rho) e^{i}px/k
\]
expansion into momentum eigenstates
Fourier transformation
The expansion into momentum eigenstates and the Fourier transformation are one and the same. Since $\psi(x)$ and $\phi(p)$ contain the same information about the particle, we can use either one to characterize the position and momentum of the particle. The more fundamental notion is the state of the particle (a state is a vector in Hilbert space), the state can be described (written down) in various representations (like position representation $\psi(x)$, momentum representation $\phi(p)$, energy representation $E$) associated with Hamiltonian operators (position $\hat{x}$, momentum $\hat{p}$, energy $\hat{E}$).

We call $\phi(p)$ the momentum representation of a particular state, and interpret it as the wavefunction in momentum space.

The SE governs the space-time evolution of the wavefunction, or equivalently, the time evolution of the state of the particle in Hilbert space.

For one particle in one (three) dimensions the Hilbert
space is one- (three-) dimensional, but for
N particles in three dimensions the Hilbert space
is 3N- dimensional. In general, it cannot be
factored into a tensor product of N three-
dimensional vector spaces \( V_{\text{sys}} \cong V_1 \otimes V_2 \otimes \ldots \otimes V_N \),
or equivalently, the wave function for N particles
does not factor into a product of wave functions
for each particle,

\[ \Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \Psi_1(\mathbf{r}_1) \Psi_2(\mathbf{r}_2) \ldots \Psi_N(\mathbf{r}_N) \]

In this case, when the wave function for an N-particle
system cannot be written as a product of wave functions
for the individual particles, i.e., when the particles
do not evolve independently, we speak of an
entangled state.

Because of this possibility a quantum system of N particles
is vastly (exponentially in N) richer than a classical
system of N particles. However, in most cases we lose
track of the particle-particle correlations associated
with entanglement, and the system behaves quasi-classically. A system that could preserve the correlations, and that could be manipulated externally, would constitute a quantum computer.

A quantum computer could solve certain computation problems 1 3 (three) have been discovered so far exponentially faster than a classical computer.

Because of the enormous size of the Hilbert space, certain quantum mechanical problems involving many-particle correlations (e.g., high-temperature superconductivity that involves correlated motion of many electrons) are very difficult to solve, or simulate on a classical computer.

Back to a single particle in one dimension: