Radial equation for given angular momentum eigenstate \( \psi_n(\theta, \phi) \) with quantum number \( \ell \)

\[
\left[ -\frac{\ell^2}{2\mu} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) + \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} + V(r) \right] R_n(r) = E_n R_n(r)
\]

can be written in form of 1D SE with effective potential \( V_{\text{eff}}(r) = V(r) + \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} \)

by defining \( U(r) = r R(r) \)

\[
-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V_{\text{eff}}(r) U(r) = E U(r)
\]

Specialization to hydrogen atom \( V(r) = -\frac{Z e^2}{4\pi \epsilon_0 r} \)

Define dimensionless variables \( p = \sqrt{\frac{8m E}{\hbar^2}} r \)

\( \lambda = \frac{2 e^2}{4\pi \epsilon_0 \hbar} \sqrt{\frac{m}{2Ze}} \)

\[ u''(p) + \left( \frac{1}{4} - \frac{1}{p^2} - \frac{\ell(\ell+1)}{p^2} \right) u(p) = 0 \]

Asymptotic solutions \( u(p) = s(p) p^\ell e^{-p/2} \)

for \( p \to \infty \), \( p \to 0 \) and Taylor expansion

\[ s(p) = \sum_{k=0}^{\infty} a_k p^k \]
leads to recursion relation
\[
\frac{a_{n+1}}{a_n} = \frac{n + l + 1 - 1}{(n+1)(n+2)(l+1)}
\]

Boundary conditions for \( p \to 0 \) require series to terminate at some \( k = \hbar r \)
\( \hbar r + l + 1 = \lambda \) \( \hbar r \) radial quantum number

Define \( \lambda = n = \hbar r + l + 1 \) principal quantum number

\[
E_n = -\frac{1}{2} m c^2 \frac{\hbar^2 r^2}{\hbar^2}
\]

not relativistic formula, only written in simple form using \( \lambda = \frac{e^2}{4\pi \epsilon_0 c} \) line structure constant
In general, the (unnormalized) polynomial \( s(p) \) is
the associated Laguerre polynomial,
\[
s(p) = \frac{e^{-p/2}}{\sqrt{2\pi p}} \int_0^\infty e^{-t} t^{n-1/2} dt,
\]
defined as
\[
L_n^\lambda (p) = \sum_{m=0}^{n} \binom{n+\lambda}{n-m} \frac{(-1)^m}{m!}.
\]

The 3D wavefunction is given by
\[
\Psi_{n\ell m}(r, \theta, \phi) = R_{n\ell}(r) Y_{\ell m}(\theta, \phi)
\]
\[
= \frac{1}{\sqrt{4\pi}} R_{n\ell}(r) Y_{\ell m}(\theta, \phi)
\]

with \( u(p) = s(p) p^{\ell+1} e^{-p/2} \)

normalized such that
\[
1 = \int d^3r |\Psi_{n\ell m}(r)|^2 = \int d\Omega |Y_{\ell m}(\theta, \phi)|^2 \int r^2 dr R_{n\ell}(r)^2 \]

The probability to find particle within shell
\([r, r + dr]\) is given by
\[
|\Psi_{n\ell m}(r)|^2 r^2 |R_{n\ell}(r)|^2 dr = u(p)^2 |dL_{n\ell m}/dp|
\]
Degeneracy of the hydrogen spectrum

for given \( l \), all magnetic quantum numbers \( m \) have the same energy, so each \( l \) is \((2l+1)\) degenerate.

Also, for each \( n = n_r + l + 1 \), the radial quantum number \( n_r \) can take on the values \( n_r = 0, 1, \ldots, n-1 \), and the \( l \) quantum number the corresponding values \( l = 0, 1, \ldots, n-1 \).

So for given \( n \) the total number of degenerate states is

\[
1 + 3 + \ldots + 2(n-1)+1 = (n+1)^2
\]

(\( l = 0 \) \( l = 1 \) \( l = n-1 \))

Actually, there are twice as many states because each electron has two spin states.
h=4 4f  l=3  4d  l=2  4p  l=1  4s  l=0

h=3 3d  l=2  3p  l=1  3s  l=0

h=2 2p  l=1  2s  l=0

n l  (l=0 is called S, l=1 is called P, l=2  D, l=3 F)

explains structure of periodic table
Normalized time-independent eigenstates of hydrogen

\[ \psi_{100} = \frac{2}{a_0^{3/2}} e^{-r/a_0} Y_{00}(\theta, \phi) \]

\[ \psi_{200} = \frac{2}{(2a_0)^{3/2}} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} Y_{00}(\theta, \phi) \]

\[ \begin{pmatrix} \psi_{211} \\ \psi_{210} \\ \psi_{210} \end{pmatrix} = \frac{1}{\sqrt{3} (2a_0)^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \begin{pmatrix} Y_{11}(\theta, \phi) \\ Y_{10}(\theta, \phi) \\ Y_{10}(\theta, \phi) \end{pmatrix} \]

\[ \text{etc} \]
\( n_r = 0 \)
\( l = 0 \)
\( n = 1 \)

\( n_r = 1 \)
\( l = 0 \)
\( n = 2 \)

\( n_r = 0 \)
\( l = 1 \)
\( n = 2 \)

\( n_r = 2 \)
\( l = 0 \)
\( n = 3 \)

\( N_r \): number of nodes in radial wavefunction

e etc
Some useful expectation values for the hydrogen atom

Given the wavefunctions $\psi_n(r)$, we can calculate expectation values $\langle r^k \rangle = \int_0^\infty r^k \psi_n^*(r) \psi_n(r) \, dr$.

$\langle r \rangle = \frac{a_o}{2^2} \left[ 3n^2 - l(l+1) \right]$

$\langle r^2 \rangle = \frac{a_o^2 n^2}{2^2} \left[ 5n^2 + 1 - 3l(l+1) \right]$

$\langle \frac{1}{r} \rangle = \frac{7}{a_o n^2}$

$\langle \frac{1}{r^2} \rangle = \frac{7^2}{a_o^2 n^3 (l+\frac{1}{2})}$

Listing of degeneracy in hydrogen atom

Further interactions, that we have neglected so far, such as the coupling of the magnetic moment associated with the spin of the electron to the magnetic field produced by the current associated with the moving electron (or proton, from the electron’s point of view) lift the degeneracy between $S, P, D...$ levels: spin-orbit interaction.
Furthermore, relativistic effects lead to energy shifts that depend on the total angular momentum $\tilde{J} = I + \tilde{S}$ of the electron (8.05: addition of angular momenta). Furthermore, the proton has spin and a small magnetic moment associated with it; the interaction between the proton's and the electron's magnetic moments is called the hyperfine interaction, and leads to shifts that depend on the total angular momentum

$$\tilde{F} = \tilde{J} + \tilde{I} = \tilde{I} + \tilde{S} + \tilde{I}$$

of the atom, where $\tilde{I}$ is the spin of the proton (nucleus).

While intrinsic angular momentum (spin) of fundamental particles is always half-integer, composite particles (e.g., nuclei) can have integer spin if the number of constituents is even. Viewed as simple particles, atoms can be bosons (integer spin) or fermions (half-integer spin).
With dramatic consequences for quantum statistics and low-temperature behavior:

Two identical fermions must be described by a wavefunction that is antisymmetric with respect to particle exchange,

$$\Psi_1(\vec{r}_1, \vec{r}_2) = -\Psi_1(\vec{r}_2, \vec{r}_1)$$

(1) wavefunction vanishes for \( \vec{r}_1 = \vec{r}_2 \) => fermions avoid each other

Bosons are described by symmetric wavefunction with respect to particle exchange

$$\Psi_0(\vec{r}_1, \vec{r}_2) = +\Psi_0(\vec{r}_2, \vec{r}_1)$$

(2) bosons are more likely to be found at same position => lasers, Bose-Einstein condensation, superconductivity, classical notion of fields where amplitudes can be added.
Polarization states of light

A classical light field, traveling along +z can be linearly polarized along x,
\[ \tilde{E}(z,t) = E_0 \hat{e}_x e^{i(kz - \omega t)} \]
linearly polarized along y,
\[ \tilde{E}(z,t) = E_0 \hat{e}_y e^{i(kz - \omega t)} \]
linearly polarized along a direction \( \hat{e} = \cos \theta \hat{e}_x + \sin \theta \hat{e}_y \) in the xy plane,
\[ \tilde{E}(z,t) = E_0 \hat{e} e^{i(kz - \omega t)} \]
circularly polarized \( \hat{e}_r = \frac{1}{\sqrt{2}} (\hat{e}_x + i \hat{e}_y), \hat{e}_l = \frac{1}{\sqrt{2}} (\hat{e}_x - i \hat{e}_y) \)
\[ \tilde{E}_{r,l}(z,t) = E_0 \hat{e}_{r,l} e^{i(kz - \omega t)} \]
or in general elliptically polarized \( \hat{e} = \cos \theta \hat{e}_x + e^{i\phi} \sin \theta \hat{e}_y \)
\[ \tilde{E}(z,t) = E. \hat{e} e^{i(kz - \omega t)} \]

Any two orthogonal polarizations (e.g., \( (\hat{e}_x, \hat{e}_y), \) \( (\frac{1}{\sqrt{2}} \hat{e}_x + \hat{e}_y), \frac{1}{\sqrt{2}} (\hat{e}_x - \hat{e}_y) \), \( (\hat{e}_r, \hat{e}_l) \), ...) form a basis in that any arbitrary polarization can be expressed as a superposition of the two basis polarizations.
A linear polarizer has one strongly absorbing direction of polarization (ideally $\vec{\varepsilon} \cdot \hat{\varepsilon} = 0$ along that direction of polarization after the polarizer), and one weakly absorbing direction (ideally, no absorption). If we call the latter the axis of the polarizer, the light behind the polarizer is linearly polarized along that axis.

\[ \vec{\varepsilon}_t = (\vec{\varepsilon}_i \cdot \hat{\varepsilon}_x) \hat{\varepsilon}_x \ e^{iut - i\omega t} \]

No light is transmitted through two crossed polarizers unless a third polarizer is inserted between them at an intermediate angle.

\[ \vec{\varepsilon}_t = \vec{\varepsilon}_i \cdot \hat{\varepsilon}_x \]

In this case the transmitted field is
\[ \vec{\varepsilon}_t \cdot (\hat{\varepsilon}_x + \frac{1}{2} (\hat{\varepsilon}_x + \hat{\varepsilon}_y)) \times \frac{1}{2} (\hat{\varepsilon}_x + \hat{\varepsilon}_y) \cdot \hat{\varepsilon}_x = \frac{1}{2} \vec{\varepsilon}_i \]
and the transmitted intensity is \( \frac{1}{4} I_0 \propto \frac{1}{2} \varepsilon_0 \).
The variation of transmitted electric field with polarizer angle, \( \hat{E} = \cos \theta \hat{e}_x + \sin \theta \hat{e}_y \), for incident field along \( \hat{e}_x \), \( \hat{E} = \hat{e}_x \hat{e}_x \) is \( \hat{e}_x \cdot \hat{E} = \cos \theta \), so the transmitted intensity varies as \( \cos^2 \theta \).

**Quantum mechanical description:**

A light beam consists of photons. If we attenuate the beam to the level where only one photon passes through the polarizer at any given time, then because photons appear only as units, the photon is either absorbed or it is not. The probability for the photon passing the polarizer is now \( \cos^2 \theta \) (the probability amplitude is \( \theta \)). The polarizer "measures" the polarization state of the photon: if the photon is polarized along the polarizer axis, it is transmitted, if polarized perpendicular to the polarizer axis, the photon is absorbed.