Angular momentum

The eigen equation associated with angular momentum reads

\[ \hat{\mathbf{L}}^2 Y(\theta, \phi) = 2m r^2 E \cdot Y(\theta, \phi) = \text{const.} \cdot Y(\theta, \phi) \]

where \( 2m r^2 E \) is the eigenvalue, and

\[ \hat{\mathbf{L}}^2 = -i \hbar \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \]

Similar to the HO problem, we can proceed in two ways:

We can either solve the differential equation using some Taylor expansion, or we can take a more abstract operator approach. Here we will do the latter. (For the direct approach see Basiorowicz, supplement 7-B, or T&T.)

We analyze the commutation relations for the angular momentum operator

\[ \hat{\mathbf{L}} = \hat{\mathbf{L}}_x \hat{x} + \hat{\mathbf{L}}_y \hat{y} + \hat{\mathbf{L}}_z \hat{z} \]

Note that since waves in orthogonal directions are independent, we have no Heisenberg uncertainty restriction on, say, \( x \) and \( p_y \), and consequently the commutator is zero, \([x, p_y] = 0\).
Let us calculate the commutator between different components of \( \mathbf{L} \): on it operator symbol
\[
\begin{align*}
[\mathbf{L}_x, \mathbf{L}_y] &= [x p_x - y p_y, y p_x + x p_y] = y [p_x, y] p_x + x [p_x, -x p_y] p_y = \\
&= \frac{i}{\hbar} p_x + i t x p_y = i t (x p_y - y p_x) = i t \mathbf{L}_z
\end{align*}
\]
\[
[\mathbf{L}_x, \mathbf{L}_y] = i t \mathbf{L}_z, \quad [\mathbf{L}_x, \mathbf{L}_z] = i t \mathbf{L}_x, \quad [\mathbf{L}_z, \mathbf{L}_x] = i t \mathbf{L}_y,
\]
This means (see p. 214) that it is not possible to choose simultaneous eigenfunctions of, say, \( \mathbf{L}_x \) and \( \mathbf{L}_y \) unless \( L_z = 0 \) for that state.

What about \( \mathbf{L}_z^2 \)?
\[
[\mathbf{L}_z^2, \mathbf{L}_z] = [\mathbf{L}_x, \mathbf{L}_z^2 + \mathbf{L}_y^2 + \mathbf{L}_z^2] = \mathbf{L}_z [\mathbf{L}_x, \mathbf{L}_z] + \mathbf{L}_x [\mathbf{L}_z, \mathbf{L}_z] = \\
= \mathbf{L}_x [\mathbf{L}_z, \mathbf{L}_z] + \mathbf{L}_z [\mathbf{L}_x, \mathbf{L}_z] \mathbf{L}_x + \mathbf{L}_y [\mathbf{L}_z, \mathbf{L}_y] \mathbf{L}_y
\]
\[
= i t \mathbf{L}_x \mathbf{L}_y + i t \mathbf{L}_y \mathbf{L}_x - i t \mathbf{L}_y \mathbf{L}_x = 0
\]
This implies that one can find simultaneous eigenstates of \( \mathbf{L}_z^2 \) and one component of \( \mathbf{L}_z \), e.g., \( \mathbf{L}_x \), but not of all components.

Proof by contradiction: For simultaneous eigenstate \( |u\rangle \)
of \( \mathbf{L}_x \) and \( \mathbf{L}_y \) with \( \mathbf{L}_x |u\rangle = \lambda_1 |u\rangle \) and \( \mathbf{L}_y |u\rangle = \lambda_2 |u\rangle \), we have \( [\mathbf{L}_x, \mathbf{L}_y] |u\rangle = 0 \), \( \lambda_2 |u\rangle \), and
\[ l_z \langle n \rangle = l_z \langle n \rangle = \frac{1}{i \hbar} [L_z, l_z] \langle n \rangle = 0 \quad \Rightarrow \quad l_z = 0 \quad \text{and} \]

Similarly \( l_z = 0 \implies \)

Only for \( l_z = 0 \) can we have simultaneous eigenstates of \( L_x, L_y, L_z \).

In general, we can only have simultaneous eigenstates of \( \ell^2 \) and \( L_z \) (or \( L_x \) or \( L_y \), \( L_z \) convention).

Let us denote such an eigenstate by \( \ell_z \langle n \rangle \) with \( L_z|\ell_z \langle n \rangle \rangle = m \ell_z|\ell_z \langle n \rangle \rangle \) and

\[ \ell^2|\ell_z \langle n \rangle \rangle = \hbar^2 \ell(\ell+1)|\ell_z \langle n \rangle \rangle \]

The reason for the strange definition of the quantum number \( \ell \) (or \( \ell^2 \) eigenvalue \( \ell(\ell+1) \)) will become apparent later. \( n, \ell \) are dimensionless numbers, since \( \ell = \hat{r} \times \hat{p} \) has units of \( \hbar \).

We assume that the simultaneous eigenstates \( \ell_z \) of \( \ell^2 \) and \( L_z \) are normalized, 

\[ \langle \ell_z \langle n \rangle |\ell_z \langle n \rangle \rangle = \delta_{\ell_z \langle n \rangle} \delta_{\ell_z \langle n \rangle} \]

orthonormality for angular momentum eigenstates.
It is useful to define the following non-Hermitian operators:

\[ L_+ = L_x + i L_y \quad \text{and} \quad L_- = L_x - i L_y \]

\[ L^+ = L_- \quad \text{and} \quad L^- = L_+ \]

(reminiscent of \( \hat{a} = \frac{\sigma}{\hbar} + i \frac{p}{\hbar} \), \( \hat{a}^+ = \frac{\sigma}{\hbar} - i \frac{p}{\hbar} \))

To understand the significance of these operators, let us analyze their commutation relations:

\[ [L^+, L^+] = 0 \]

since \( [L^+, L_x] = 0 \), \( [L^+, L_y] = 0 \)

\[ [L_+, L_-] = [L_x + i L_y, L_x - i L_y] = -i [L_x, L_y] + i [L_y, L_x] = -2i \quad \text{and} \quad [L_x, L_y] = -2i \quad \text{and} \quad [L_y, L_x] = 2i \]

\[ [L_+, L_-] = 2i \quad \text{and} \quad [L_+, L_-] = 2i \]

\[ [L_+, L_+] = [L_x + i L_y, L_x + i L_y] = [L_x L_x + i L_x L_y + i L_y L_x + L_y L_y] = 2L_x L_x + 2i L_y L_x = 0 \]

\[ [L_-, L_-] = 2L_x \]

\[ [L_+, L_x] = [L_x + i L_y, L_x] = [L_x, L_x] + i [L_y, L_x] = 0 + i L_y = i L_y \]

\[ [L_-, L_+] = \mp i L_x \]

We also note that:

\[ L_+ L_- = (L_x + i L_y)(L_x - i L_y) = L_x^2 + L_y^2 - i L_x L_y + i L_y L_x = L_- L_+ - i [L_x L_y] \]

\[ = L_- L_+ + t L_z \]
and similarly \[ L_+ L_- = L_x^2 - L_z^2 + t L_2 \]
\[ L_- L_+ = L_x^2 - L_z^2 - t L_2 \]

As for the HO, we now proceed to analyze the range of allowed values for \( l, m \):

Since \( L^2 = L_x^2 + L_y^2 + L_z^2 \) and \( L_x, L_y, L_z \) are Hermitian operators, we have

\[ \langle l, m | L_x^2 | l, m \rangle = \langle L_x | L_x | l, m \rangle = \langle L_x | L_{x} | l, m \rangle = 2 \langle L_{x} | L_{x} | l, m \rangle \geq 0 \]

Similarly for \( L_y^2 \) and consequently \( \langle l, m | L^2 | l, m \rangle \geq 0 \)

or \( 0 \leq \langle l, m | L^2 | l, m \rangle = t^2 \langle l+1, l+1 | l, m \rangle = t^2 \delta(\ell+1) \)

Consequently, we can choose \( l \geq 0 \) \( \left[ \text{If} \quad l' \leq -1, \quad \text{we define} \quad \ell' := -\left(l+1\right) \quad \text{then} \quad \ell \left(l+1\right) = -\ell' \left(l'+1\right) \quad \text{and} \quad l' \geq 0 \right] \)

To understand the operators \( L_x, \) let us define a new state \( |\psi_2\rangle := L_+ |l, m\rangle \), and act on it with \( L^2 \)

\[ L^2 |\psi_2\rangle = L_x^2 |l, m\rangle = L_+ C^2 |l, m\rangle = t^2 \ell (\ell + 1) L_+ |l, m\rangle = t^2 \delta(\ell+1) |\psi_2\rangle \]

so \( |\psi_2\rangle \) is an eigenstate of \( L^2 \) with the same quantum number \( l \)
Also we have
\[
\langle l', \ell'_{1+n} | \ell_{m+n} \rangle = \langle l', l_{2 \pm t} | l_{m} \rangle = \langle l'_{1+n} \pm t | l_{2} \rangle \langle l_{m} \rangle = (n+1) \langle l_{+1} | l_{m} \rangle = (n+1) \ell_{1+n}
\]
This means that \( \ell_{m+n} \) is also an eigenstate of \( \ell_{+} \), but with an eigenvalue \((n+1)\ell_{+} \) that differs from the original one by one. Since \( n \) is the quantum number associated with the \( z \)-component of angular momentum, we call \( n \) the azimuthal (or magnetic) quantum number, while \( \ell \) is the quantum number associated with total angular momentum.

\( \ell_{+} \) (\( \ell_{-} \)) raises (lowers) the magnetic quantum number by one, while preserving the total angular momentum. Let us calculate the length of \( \langle \ell_{m+n} | \ell_{m+n+1} \rangle = \langle \ell_{m} | \ell_{m+1} \rangle \)
the unnormalized state vector.

\[
\langle \ell_{m+1} | \ell_{m+n+1} \rangle = \langle \ell_{m} | \ell_{m+1} \rangle = \langle \ell_{m} | \ell_{2 \pm t} \rangle \langle \ell_{m+1} \rangle = \langle \ell_{m} | l_{+1} \rangle \langle \ell_{m+1} \rangle = (t^{2} l(l+1) - t^{2} m^{2} + t^{2} m) \langle \ell_{m} | \ell_{m+1} \rangle = t^{2} (l(l+1) - m(m+1))
\]

\[
= t^{2} (l^{2} m)(l^{2} m + 1)
\]
Since the length squared of any vector must be non-negative, it follows that

\[ l^2 + 1 - m^2 (m + 1) \geq 0 \]

Consequently, 

\[ m^2 = m + 1 \implies \frac{m+1}{4} - \frac{1}{4} \leq l^2 \implies l^2 = (l + \frac{1}{4}) - \frac{1}{4} \]

or 

\[ |l + \frac{1}{4}| \leq |l + \frac{1}{2}| = l + \frac{1}{2} \quad \text{since} \quad l \geq 0 \]

\[ \implies m \leq l \quad \text{for} \quad m > 0 \quad \text{and also} \quad -m \leq l \quad \text{for} \quad m \leq 0 \]

Therefore, \( m \) is bounded both from above and from below:

\[ -l \leq m \leq l, \quad l \geq 0 \]

Since \( |\psi_+\rangle = |l+1/2m\rangle \) is also an eigenstate of \( L^2 \) and \( L_z \), but with new eigenvalue \( m' = m + 1 \), the bound on \( m \) is only consistent with this fact if \( L^2 |l+1/2m\rangle = 0 \) for some \( m \). Consequently, with \( L^2 |l+1/2m\rangle = |l+1/2m+1\rangle \)

\[ 0 = \langle l+1/2m| l+1/2m+1 \rangle = h^2 (l - m) (l + m + 1) \]

\[ \implies m_{\max} = l \]

Similarly, for \( |\psi_-\rangle = |l-1/2m\rangle \) we have

\[ m_{\min} = -l \]
Thus we have a ladder of eigenvalues spaced by one, and connected by the raising and lowering operators $L^+$ and $L^-$. 

\[ n = -l, -l+1, \ldots, l-1, l \]

\[ l \geq 0 \]

\[ L^+ | n \rangle = \sqrt{n+1} | n+1 \rangle \]
\[ L^- | n \rangle = \sqrt{n} | n-1 \rangle \]
\[ L^+ | l \rangle = \sqrt{l+1} | l+1 \rangle \]
\[ L^- | l \rangle = \sqrt{l} | l-1 \rangle \]

This is only possible if $l$ is integer or half integer.

It turns out that half-integer values of $l$ have no simple spatial representation, and correspond to an internal form of angular momentum called spin of the particle. Here we will restrict ourselves to orbital angular momentum, which requires $l$ to be an integer.