The momentum eigenstates in the position representation, \( u_p(x) \), defined by

\[
\hat{p} u_p(x) = \frac{i \hbar}{m} \frac{d}{dx} u_p(x) = p u_p(x)
\]

and given by

\[
u_p(x) = \frac{1}{\sqrt{2\pi \hbar}} e^{ipx/\hbar} \]

cannot be normalized in free space to be interpreted as a probability density since \( |u_p(x)|^2 = \frac{1}{2\pi \hbar} \), and \( \int_{-\infty}^{\infty} |u_p(x)|^2 \) diverges. However, they do satisfy the continuum orthogonality condition:

\[
\int_{-\infty}^{\infty} u_p^*(x) u_{p'}(x) dx = \delta(p-p')
\]

This normalization corresponds to a uniform particle density (particles per meter) given by: \( |u_p(x)|^2 = \frac{1}{2\pi \hbar} \).

Let us calculate the probability current (particle moving past a point \( x \) per second), defined by

\[
 j(x) = \frac{\hbar}{2im} \left[ \psi^*(x) \frac{d\psi}{dx}(x) - \left( \frac{d\psi^*(x)}{dx} \right) \psi(x) \right]
\]

For \( \psi(x) = u_p(x) \) we find

\[
 j(x) = \frac{\hbar}{2im} \frac{1}{2\pi \hbar} \left[ \frac{p}{m} - \left( \frac{ip}{m} \right) \right] = \frac{1}{2\pi \hbar m} \]

which is exactly what we expect for a uniform particle density \( |u_p(x)|^2 = \frac{1}{2\pi \hbar} \) moving at velocity \( v = \frac{p}{m} \).
In general, choosing a wavefunction \( \psi(x) = Ce^{ix} \) corresponds to particles moving at velocity \( \frac{p}{m} \), a particle density \( \langle \psi(x) \rangle^2 = |C|^2 \), and a particle current \( j(x) = |C|^2 \). Alternatives do deal with normalization problem (wavefunction not square-integrable) for momentum states:

a) Wavepackets

A superposition of a finite number of momentum eigenstates is not normalizable, but a wavepacket consisting of an infinite number of momentum eigenstates (Fourier components) is.

\[ \int |\psi(x)|^2 dx = \int dp |\phi(p)|^2 = 1 \]

\( \psi(x) \) (\( \phi(p) \)) goes to 0 faster than \( \frac{1}{\sqrt{x^2}} \) for \( x \to \pm \infty \) (\( p \to \pm \infty \)).
6) Periodic boundary conditions

Assume box of finite length $L$,
require periodic boundary conditions

$$\psi(0) = \psi(L)$$

For plane waves $e^{i px/k}$ this implies that $e^{i pl/k} = 1$ or $pl/k = n 2\pi$, $n$ integer,
i.e. momentum is quantized,

$$p_n = n \hbar k_0 \quad \text{with} \quad k_0 = \frac{2\pi}{L}.$$ 

The corresponding momentum states are normalizable
in the interval $[0,L]$,

$$\int_0^L |C e^{i px/k}|^2 dx = |LC|^2 = 1$$

$$U_{p_n}(x) = \frac{1}{\sqrt{L}} e^{i p_n x / k_0}$$

(normalized momentum eigenstates)

in box of size $L$, with $p_n = n \hbar k_0$.

$$\int_0^L U_{p_n}^*(x) U_{p_m}(x) dx = \delta_{nm} \quad \text{with normality condition in box}$$

We perform all calculations for fixed size box,
then take the limit $L \to \infty$ (i.e. $k_0 \to 0$, momentum spectrum becomes continuous). All physically sensible results will be independent of box size $L$ as long as $L$ is large compared to distances of interest.
Time evolution of free-particle wavepackets

In free space we often work with normalized Gaussian wavepackets

\[ \Psi(x, t=0) = \frac{1}{(2\pi)^{1/4} W_0^{3/2}} e^{-\frac{x^2}{4W_0^2}} \]

Written in this form we have

\[ |\Psi(x, 0)|^2 = \frac{1}{(2\pi)^{1/4} W_0} e^{-\frac{x^2}{2W_0^2}}, \quad \int dx |\Psi(x, 0)|^2 = 1, \]

\[ \langle x \rangle = 0, \quad \langle x^2 \rangle = W_0^2, \quad \langle \Delta x \rangle = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = W_0 \]

\( \Delta x = W_0 \) is the uncertainty, or rms width (root-mean-square width) of the wavepacket.

Why this form of wavepacket?

1) Particularly simple and symmetric, the Fourier transform is also a Gaussian wavepacket

\[ \tilde{\Phi}(k) = \frac{1}{(2\pi)^{1/4} W_0} e^{-\frac{k^2}{4W_0^2}} \quad \text{with} \quad k_0 = \frac{\hbar}{2W_0} \]

\( \langle \Delta k \rangle = \langle k^2 \rangle - \langle k \rangle = k_0 \)

2) This is a wavepacket with the minimum uncertainty \( \Delta x \Delta k = \frac{1}{2} \) (\( \Delta x \Delta p = \frac{\hbar}{2} \)) allowed by QM
3) Physical system often give rise to Gaussian broadening in momentum or position, e.g. thermal distribution of atomic momenta in a gas is a Gaussian distribution.

How do we make a wavepacket move at velocity \( v \)?

We displace the distribution in momentum space from

\[
\langle p \rangle = \langle t * h \rangle = 0 \quad \text{to} \quad \langle p \rangle = \langle t * h \rangle = t \cdot k_0 = n \cdot v,
\]

\[
\bar{\Phi}_\epsilon(k) = \frac{1}{(2\pi)^{3/2} \hbar^{3/2} v_0} e^{-\frac{(k - k_0)^2}{4 \hbar^2}}
\]

The inverse Fourier transform is

\[
\Psi_\epsilon(x) = \frac{1}{(2\pi)^{3/4} \hbar^{3/4} v_0} e^{-\frac{x^2}{4 \hbar^2}} e^{ik_0 x}
\]

The phase variation \( e^{i k_0 x} \) of the wavefunction in position space 'encodes' the notion of the wavepacket at velocity \( v_1 = \frac{t \cdot h}{m} \).
How does a free-space Gaussian wave packet evolve in time?

In general, we expand a wavefunction \( \Psi(x,0) \) into energy eigenfunctions \( \psi_n(x) \), and then evolve each energy eigenfunction as \( e^{-iEt/\hbar} \).

In free space, there is only KE, and a wave will defined momentum \( p \) (a momentum eigenstate \( \psi_p(x) \)), has a sharply defined energy \( E_p = \frac{p^2}{2m} \). We say that in free space, the momentum eigenstates \( \psi_p(x) \) are simultaneous eigenstates of energy:

\[
\hat{H} \psi_p(x) = \frac{\hbar^2}{2m} \psi_p(x) = \frac{i}{\hbar} \frac{\hbar^2}{2m} \frac{1}{\sqrt{2\pi\hbar}} e^{i}px/\hbar = \frac{p^2}{2m} \psi_p(x)
\]

or \( \hat{H} \psi_p(x) = \frac{p^2}{2m} \psi_p(x) = E_p \psi_p(x) \) in free space.

The energy eigenstates are said to be doubly degenerate: for each eigenvalue of energy \( E > 0 \) there are two different momentum states (namely \( \psi_{\pm p}(x) \) with \( p = \sqrt{2mE} \)) that have the same energy.
It follows that a momentum eigenstate with eigenvalue $p$ evolves in time as $e^{-ip\cdot t/t}$, so that the wavefunction in momentum space evolves in time as

$$\Psi(p, t) = \Phi(p, 0) e^{-i\frac{p^2}{2m}t/t}$$

The wavefunction in real space is given by the inverse Fourier transform of $\Psi(p, t)$, or equivalently, as the superposition of energy eigenfunctions with their corresponding phase evolution factors $e^{-iE_p t/t}$:

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi t}} \int dp \, \Psi'(p, t) e^{i px/t} = \int dp \, \Phi(p, t) U_p(x)$$

$$= \int dp \, \Phi(p, 0) U_p(x, t) = \int dp \, \Phi(p, t) U_p(x)$$

where $U_p(x, t) = U_p(x) e^{-i\frac{p^2}{2m}t/t} = \frac{1}{\sqrt{2\pi t}} e^{i px/t} e^{-i\frac{p^2}{2m}t/t}$ are the time-dependent momentum eigenfunctions in free space.

The above equation shows that the phases of different
Fourier components \( u_n(x) = \frac{1}{\sqrt{2\pi}} e^{i p x / \hbar} \) evolve in time at different speeds, the "running out of phase" of different Fourier components leads to a spreading of the wavepacket in position space. In the problem sets you will show that the RMS width \( \Delta x(t) = W(t) \) of the wavepacket grows in time as

\[
W(t) = W_0 \sqrt{1 + \frac{t^2 \Delta E^2}{m^2 \hbar^2}}
\]

Since a wavepacket contains different momentum components, it changes in time in free space even though there are no external forces acting.

For long times \( t \gg t_0 = \frac{m \hbar}{E_0} \) the wave packet spreads as \( W(t) \approx \frac{t}{m W_0} \), i.e. at a speed \( v_0 = \frac{t}{m W_0} \) that is inversely proportional to its initial size. That speed is negligible for macroscopic wavepacket size, but can be appreciable for initially well-localized microscopic objects.
The spreading of a wave packet in free space was early evidence that the wave packet size cannot be identified with the particle size. The spreading is due to the quadratic (i.e., not linear) dependence of the energy, and hence the phase evolution rate, on momentum. Note that the wave packet of a massless particle, e.g., a photon, with \( E = pc \) would not spread. (The SC is non-relativistic and does not apply to photons.)

Motion of wave packets, group velocity, and stationary phase

Why is it that a wave function

\[ \Psi(x, t) = \frac{1}{(2\pi)^{3/2} \nu_0} e^{-\frac{x^2}{4\nu_0^2}} e^{i\nu_0 x - \frac{iE}{\hbar} t} \]

represents a particle moving at velocity \( v = \frac{\nu_0}{m} \)?

The velocity of a crest moving forward for a simple momentum component \( u_0(x, t) = \frac{1}{\nu_0} e^{ik_0 x - \frac{iE}{\hbar} t} \) is (crest moves forward distance \( \lambda = \frac{2\pi}{k_0} \))
time \[ T = \frac{2\pi}{\omega} \text{ where } \omega = \frac{\beta k_1}{2m} \quad (e^{-i\omega t} = e^{-iut}) \]

\[ V_{ph} = \frac{\lambda}{T} = \frac{2\pi}{\omega} \frac{\omega}{2\pi} = \frac{\omega}{\lambda} = \frac{\beta k_1}{2m} \]

phase velocity of a momentum component

The particle does not move at the phase velocity \( V_{ph} \)
\[ V_{ph} = \frac{\omega}{k_1} \]
which is the plane wave associated with a single momentum moves forward.

At what velocity then?
look at exponent and write \[ \xi = E(k_1) = tw_1 + tw(k_1) = \frac{\beta k_1^2}{2m} \]

\[ \exp\left(-\frac{x^2}{4w_1^2} + ik_1x - iw(k_1)t\right) \]

Remember Fermat's principle of stationary phase:

path is defined by region of space where phasors point mostly in one direction, i.e. where the phase \[ \phi(x) = -\frac{x^2}{4w_1^2} + ik_1x - iw(k_1)t \] does not vary between different momentum components \( k_1 \) to lowest order.

\[ 0 \equiv \frac{\partial \phi}{\partial k_1} = ix - i\frac{\partial \omega}{\partial k_1}t = i\left(x - \frac{\partial \omega}{\partial k_1}t\right), \text{ or} \]

\[ x_{ce}(t) = \left(\frac{\partial \omega}{\partial k_1}\right)t \]
Fermat's principle leads us to the concept of group velocity.

\[ \nu_{gr} = \frac{\partial \omega}{\partial k} = \frac{k}{m} = \frac{p}{m} \]

This is the group velocity of a wave packet at which the wave packet, i.e., the region of constructive interference, propagates.

The difference between group and phase velocity is due to the fact that \( \frac{\partial \omega}{\partial k} \neq \frac{\omega}{k} \) or \( \frac{\partial E}{\partial p} = \frac{\partial (tk)}{\partial (tk)} \neq \frac{E}{p} \)

i.e., the quadratic dependence of \( ku \) on momentum in free space.